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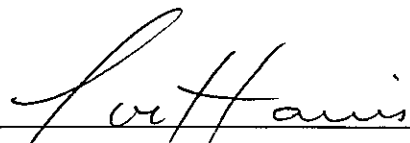Graduate School of Arts and Sciences

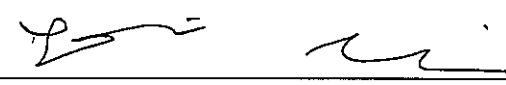DISSERTATION ACCEPTANCE CERTIFICATE

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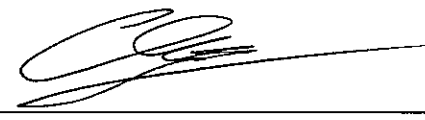
have examined a dissertation entitled

Stable log surfaces, trigonal covers, and canonical curves of genus 4

presented by **Changho Han**

candidate for the degree of Doctor of Philosophy and hereby
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Date: April 23, 2019

Stable log surfaces, trigonal covers, and canonical curves of genus 4

A dissertation presented

by

Changho Han

to

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Stable log surfaces, trigonal covers, and canonical curves of genus 4

Abstract

We describe a compactification of the moduli space of pairs (S, C) where S is iso-morphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and $C \subset S$ is a genus 4 curve of class $(3, 3)$. We show that the compactified moduli space is a smooth Deligne-Mumford stack with 4 boundary components. We relate our compactification with compactifications of the moduli space $\mathcal{M}_4$ of genus 4 curves. In particular, we show that our space compactifies the blow-up of the hyperelliptic locus in $\mathcal{M}_4$. We also relate our compactification to a compactification of the Hurwitz space $\mathcal{H}_4^3$ of triple coverings of $\mathbb{P}^1$ by genus 4 curves.

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CONTENTS

I owe special thanks to my family, who continuously cheered me up and encouraged me to continue on mathematics. Special thanks to my parents who taught me to never give up and move forward.

1. Introduction

Algebraic geometry originated in the study of zero sets of polynomials. An important question in algebraic geometry is about the classification of projective varieties. It is remarkable that certain collections of projective varieties are themselves parametrized by points of algebraic varieties, called *moduli spaces*. A well-known example is the moduli space $\mathcal{M}_g$ of smooth projective curves of a fixed genus g. In general, a point of a moduli space $\mathcal{M}$ corresponds to an object in the classification, and a map from a space V to $\mathcal{M}$ corresponds to a family of objects parametrized by V. In this way, $\mathcal{M}$ not only gives ways to understand individual objects, but also naturally gives ways to understand families of objects.

A crucial step in the study of moduli spaces is the notion of a modular compactification. A moduli space $\overline{\mathcal{M}}$ is a *modular compactification* of a moduli space $\mathcal{M}$ if $\overline{\mathcal{M}}$ is compact and contains $\mathcal{M}$ as a dense open subset. Since understanding compact spaces is much easier than noncompact spaces, it is useful to have a modular compactification. Note that moduli spaces of smooth varieties usually are not compact. To compactify them, one can enlarge the class of smooth varieties by allowing singularities. If we carefully select singularity conditions on varieties, then we often get a compact moduli space. For example, enlarging the class of smooth projective curves by adding stable curves of genus $g \geq 2$, Deligne and Mumford in [11] constructed a compact moduli space $\overline{\mathcal{M}}_g$ of stable curves as a modular compactification of $\mathcal{M}_g$. The discovery of $\overline{\mathcal{M}}_g$ led to significant advances not only in algebraic geometry, but also in related fields such as number theory and mathematical physics. For instance, Harris and Mumford showed in [21] that the moduli space $\mathcal{M}_g$ is of general type for $g \geq 24$ by using divisors on $\overline{\mathcal{M}}_g$, Kontsevich used intersection theory on $\overline{\mathcal{M}}_g$ to give an algebraic formulation of Gromov-Witten theory (see [16] for a survey), and Deligne and Mumford gave an algebraic proof in [11] of the irreducibility of $\mathcal{M}_g$ in any positive characteristics, by using the existence of $\overline{\mathcal{M}}_g$.

In contrast to the geometry of the moduli spaces of curves, the geometry of moduli spaces of surfaces are generally unknown. Even constructing a compact moduli space of surfaces is a recent result by Kollár, Shepherd-Barron, and Alexeev [31, 6] (see [29] for a survey). Unlike the case of curves, we do not know the boundary members for compact moduli spaces of surfaces. In fact, the geometry of moduli spaces of surfaces satisfies "Murphy's law", i.e. it can get arbitrarily bad. The broad goal of the work is to identify and describe compact moduli spaces of surfaces which are well-behaved.

In particular, the goal of this thesis is to describe a compact moduli space $\mathfrak{X}$ that lies at the cusp of three different areas of study of moduli spaces in algebraic geometry, namely (1) the study of compact moduli spaces of surfaces of log general type, (2) the study of the birational geometry of the moduli space of curves, and (3) the study of alternative compactifications of Hurwitz spaces parametrizing branched coverings.

The moduli space $\mathfrak{X}$ is defined as follows. Consider a pair (S, D), where $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $D \subset S$ is a smooth divisor of class $(3, 3)$. Observe that for all $w > 2/3$, the pair (S, wD) is a surface of log general type. Set $w = 2/3 + \epsilon$, where $0 < \epsilon \ll 1$. Then $\mathfrak{X}$ is a compactification of the space of pairs (S, wD) constructed by Kollár–Shepherd-Barron/Alexeev [31, 6] (the idea of taking ϵ close to 0 is inspired by the work of Hacking [20]). The compactification $\mathfrak{X}$ parametrizes stable semi log canonical pairs of log general type. We recall the definition in detail in the main text. For now, it suffices to say that it is the analogue in higher dimensions of the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$.

Having described the space $\mathfrak{X}$, let us explain why it is remarkable from the three different points of views mentioned in the opening sentence. In sharp contrast to the case of curves, it is rare to have a complete description of the boundary of the KSBA compactification of surfaces. Furthermore, and again in contrast to the case of curves, the KSBA compactification is usually highly singular, even reducible with components of unexpected dimensions. Nevertheless, bucking the general expectations, we are able to give an explicit description of all the boundary points of $\mathfrak{X}$. Moreover, $\mathfrak{X}$ turns out to

be quite well-behaved. Denote by $\mathcal{X}^\circ$ the open substack of $\mathcal{X}$ that parametrizes (S, wD) with $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $D \subset S$ smooth of type $(3,3)$. We show the following.

Theorem 1. *The weighted KSBA compactification $\mathcal{X}$ is an irreducible and smooth Deligne–Mumford stack over $\mathbb{K}$. The closed substack $\mathcal{X} \setminus \mathcal{X}^\circ$ is the union of 4 irreducible divisors.*

We label the 4 boundary components $\mathfrak{Z}_0$, $\mathfrak{Z}_2$, $\mathfrak{Z}_4$, and $\mathfrak{Z}_{3,3}$. The log surfaces parameterized by their generic points are as follows.

$\mathfrak{Z}_0$: S is a smooth quadric surface in $\mathbb{P}^3$ and $D \subset S$ is a generic singular curve of bi-degree $(3,3)$.

$\mathfrak{Z}_2$: S is an irreducible singular quadric surface in $\mathbb{P}^3$ and D is a complete intersection of S and a cubic surface in $\mathbb{P}^3$.

$\mathfrak{Z}_4$: S is a $\mathbb{Q}$-Gorenstein smoothing of the A_1 singularity of $\mathbb{P}(9,1,2)$ and D is a smooth hyperelliptic curve away from the singular point of type $\frac{1}{9}(1,2)$.

$\mathfrak{Z}_{3,3}$: S is a union $\mathrm{Bl}_u \mathbb{P}(3,1,1) \cup \mathrm{Bl}_v \mathbb{P}(3,1,1)$ along a $\mathbb{P}^1$ and D is a nodal union of two non-Weierstrass genus 2 tails. Here, the blowups are along curvilinear subschemes u, v of length 3 (see § 6.6 for a more precise description).

We highlight some facts about the surfaces and the curves appearing at the boundary, referring the reader to § 6.6 for the complete list. There are only 8 isomorphism classes of such surfaces S, 4 of which are irreducible, and among the irreducible surfaces, 1 is non-toric. The curves D are reduced, and only have A_n singularities for $n \leq 4$.

We now come to the second facet of $\mathcal{X}$, namely its relationship to the birational geometry of $\mathcal{M}_g$. Let $\mathcal{X}_0 \subset \mathcal{X}$ be the open substack that parametrizes pairs (S, wD) with smooth D. We have a forgetful morphism

$$\mu \colon \mathcal{X}_0 \to \mathcal{M}_4.$$

Denote by $\mathcal{H} \subset \mathcal{M}_4$ the closed substack that parametrizes hyperelliptic curves. We show the following.

Theorem 2. *The forgetful map $\mu\colon \mathfrak{X}_0 \to \mathcal{M}_4$ induces an isomorphism*

$$\mathfrak{X}_0 \cong \mathrm{Bl}_{\mathcal{H}}\, \mathcal{M}_4.$$

Thus, $\mathfrak{X}_0$ provides a modular interpretation of the blowup of the hyperelliptic locus in $\mathcal{M}_4$. The map $\mu\colon \mathfrak{X}_0 \to \mathcal{M}_4$ does not extend to a regular map from $\mathfrak{X}_0$ to any known modular compactification of $\mathcal{M}_4$ (see Proposition 8.13 for a more precise statement). It does, however, extend to a morphism from $\mathfrak{X}$ to the (non-separated) moduli stack of Gorenstein curves. It would be interesting to know if the image of $\mathfrak{X}$ in this stack is a modular compactification of $\mathcal{M}_4$ in the sense of [15], or in some other sense.

We now discuss the connection of $\mathfrak{X}$ with the third area mentioned before, the alternate compactifications of Hurwitz spaces. Recall that the Hurwitz space $\mathcal{H}_g^d$ is the moduli space of maps $\phi\colon C \to P$, where C is a smooth genus g curve, P is isomorphic to $\mathbb{P}^1$, and ϕ is a finite map of degree d with simple branching. From general structure theorems of finite coverings, we know that the map ϕ gives an embedding $C \subset \mathbb{P}E$, where E is the so-called Tschirnhausen bundle of ϕ defined by $E^\vee = \phi_*\mathcal{O}_C/\mathcal{O}_P$. For a general $[\phi] \in \mathcal{H}_4^3$, we have $E \cong \mathcal{O}(3) \oplus \mathcal{O}(3)$, and hence $\mathbb{P}E \cong \mathbb{P}^1 \times \mathbb{P}^1$. We thus get a rational map $\mathcal{H}_4^3 \dashrightarrow \mathfrak{X}$ defined by the rule $\phi \mapsto (S, C)$, where $S = \mathbb{P}E$. It is not too difficult to see that this rational map extends to a regular map $\mathcal{H}_4^3 \to \mathfrak{X}$.

At the heart of our analysis of $\mathfrak{X}$ is to find a compactification of $\mathcal{H}_4^3$ on which the map $\mathcal{H}_4^3 \to \mathfrak{X}$ extends to a regular, and hence surjective, map. Unfortunately, the standard admissible cover compactification $\overline{\mathcal{H}}_4^3$ of $\mathcal{H}_4^3$ lacks this property. We appeal to an alternate compactification $\overline{\mathcal{H}}_4^3(1/6 + \epsilon)$ constructed in [12]. This compactification parametrizes weighted admissible covers $\phi\colon C \to P$. Roughly speaking, these are finite maps from a reduced curve C of arithmetic genus 4 to a nodal curve P of arithmetic genus 0 which are admissible over the nodes in the sense of Harris–Mumford [21] and where the pointed curve $(P, \mathrm{br}\,\phi)$ is $(1/6 + \epsilon)$-stable in the sense of Hassett [24]. The following theorem is the major step towards understanding $\mathfrak{X}$.

Theorem 3. *The map $\mathcal{H}_4^3 \to \mathfrak{X}$ extends to a regular map $\overline{\mathcal{H}}_4^3(1/6 + \epsilon) \to \mathfrak{X}$.*

The existence of the regular map $\overline{\mathcal{H}}_4^3(1/6 + \epsilon) \to \mathfrak{X}$ is crucial for our analysis of $\mathfrak{X}$, and occupies the technical heart of the paper. Thanks to this map, we obtain an explicit description of pairs parametrized by $\mathfrak{X}$ using the knowledge of the points of $\overline{\mathcal{H}}_4^3(1/6+\epsilon)$. This description allows us to understand the connection between $\mathfrak{X}_0$ and $\mathcal{M}_4$, leading to Theorem 2. It also allows us to directly verify that the $\mathbb{Q}$-Gorenstein deformations of the pairs we encounter are unobstructed, leading to Theorem 1.

In broad strokes, the proof of Theorem 3 goes as follows. General structure theorems of triple coverings allow us to associate to a weighted admissible cover $\phi : C \to P$ a pair (S, D), where S is a $\mathbb{P}^1$-bundle over P and $D \subset S$ is a divisor of relative degree 3 closely related to C (the curve D differs from C only if C has non-Gorenstein singularities). It turns out that the pair (S, wD) is always semi-stable, but not necessarily stable. That is, it has slc singularities, but $K_S + wD$ is not necessarily ample. Nevertheless, we show that there exists a unique stable replacement $(\overline{S}, \overline{D})$ for (S, D). That is, from (S, D) we construct a stable pair $(\overline{S}, \overline{D})$ and show that any allowable one-parameter family with central fiber (S, D) can be transformed into an allowable family with central fiber $(\overline{S}, \overline{D})$ and isomorphic to the original family away from the central fiber. These transformations involve running an appropriate minimal model program on the total space of the family. To obtain an explicit description of $(\overline{S}, \overline{D})$, we perform an explicit sequence of blow-ups and blow-downs. The birational geometry of threefolds involved in this process may be of independent interest.

Having outlined the contents of the paper, we describe previous work of Hassett and Hacking that inspired and guided us.

In [22], Hassett described the KSBA compactification of the moduli space of (S, D), where S is isomorphic to $\mathbb{P}^2$ and $D \subset S$ is a smooth quartic curve. In this case, the natural map from the KSBA compactification to $\overline{\mathcal{M}}_3$ turns out to be an isomorphism. Observe that for a quartic curve, the embedding in $\mathbb{P}^2$ is the canonical embedding. The

next case where the canonical embedding of a curve lies naturally on a surface is the case of genus 4 curves treated in this paper.

In [20], Hacking described KSBA compactifications of weighted pairs (S, D), where S is again isomorphic to $\mathbb{P}^2$ and $D \subset S$ is a smooth plane curve of degree d. Hacking's insight was to consider weighted pairs (S, wD) that are "almost K3", namely such that $K_S + wD$ is positive, but very close to 0. We have followed the same approach in this paper. The tractable description of the resulting moduli space in both Hacking's and our case suggests that it may be possible to generalize the picture to almost K3 log pairs for other del Pezzo surfaces. We are currently investigating this direction.

1.1. **Outline.** The paper is organized as follows. Section 2 is a collection of preliminary facts in the minimal model program which will be used subsequently in later sections. Section 3 recalls fundamental results about the moduli of stable log surfaces. We focus particularly on the case of almost K3 log surfaces, where it is possible to give a functorial description of the moduli stack. Section 4 describes the construction of a log surface from a triple covering of curves. It culminates in an explicit description of the pairs obtained from triple coverings $C \to \mathbb{P}^1$ where C is a genus 4 curve. Section 5 is devoted to a construction of two kinds of threefold flips that are necessary for the stable reduction of the surface pairs obtained in Section 4. Section 6 uses the flips of Section 5 to carry out stable reduction for the unstable pairs. As a result, by the end of this section, we obtain a list of the log surfaces parametrized by $\mathfrak{X}$. Section 7 shows that the $\mathbb{Q}$-Gorenstein deformation space of the pairs parametrized by $\mathfrak{X}$ is smooth. Section 8 relates $\mathfrak{X}$ to $\mathcal{M}_4$ and $\mathcal{H}_4^3$.

1.2. **Conventions.** All schemes and stacks are locally of finite type over an algebraically closed field $\mathbb{K}$ of characteristic 0. The projectivization of a vector bundle is the space of one dimensional quotients. We go back and forth between Weil divisors and the associated divisorial sheaves without comment.

2. Preliminaries on the minimal model program

In this section, we list basic facts on the minimal model program, which is a key tool in understanding the various constructions in the paper. The minimal model program, MMP in short, is concerned with the classification of projective varieties up to a birational equivalence.

Unlike the case of curves, birational equivalence is a natural condition for classification of higher dimensional varieties, due to the presence of blowup morphisms. For example, one can always blow up a smooth point p of a projective surface S to obtain a new surface $\mathrm{Bl}_p S$, which is isomorphic to S away from the exceptional divisor E of the blowup. Hence, understanding the birational equivalence class of $\mathrm{Bl}_p S$ reduces to finding exceptional curves of $\mathrm{Bl}_p S$ that can be blown down to produce the simpler surface S. By repeating this procedure, we obtain the minimal model of S, a surface that cannot be blown down. Therefore, MMP for surfaces is equivalent to the classification of minimal surfaces.

In general, a normal projective variety X is minimal if the canonical divisor K_X is nef, and is canonical if K_X is ample. Since the blowdowns above can be interpreted as a contraction of a curve E in $\mathrm{Bl}_p S$ that has a negative intersection with $K_{\mathrm{Bl}_p S}$, the minimal model program can be naively thought of as repeatedly finding curves to contract in order to turn the canonical divisor into a nef divisor. This is successful for smooth projective surfaces S, i.e. the minimal model of S can be obtained by repeated blowdowns, and the minimal model is also a smooth projective surface.

However, in higher dimension, contracting a curve E in X with $E \cdot K_X < 0$ often introduces singularities on the resulting variety. Thus, we have to understand what are the singularity types that appear under this procedure, and how to still extend the idea of contracting curves under such singular varieties. § 2.1 explains the singularity types that are considered in the MMP (e.g. singularities coming from contracting K_X-negative curves). Then, we explain the actual procedure of MMP in § 2.2. This makes use of the

singularity types discussed earlier. In § 2.3, we extend the idea of obtaining a birational model with K_X nef to K_X ample.

2.1. **Singularities of the MMP.**

Before we dive into understanding singularities of varieties, let's recall some conventions on divisors. A *Weil divisor D* on X is a formal $\mathbb{Z}$-linear combination of irreducible pure codimension 1 subvarieties of X. An *effective* Weil divisor is one where all the coefficients are non-negative. If X is normal, then Weil divisors are Cartier in codimension 1. That is, there exists an open subset $U \subset X$ whose complement is of codimension at least 2 such that the restriction of the divisor to U is Cartier. A (generically Cartier) Weil divisor D defines a reflexive sheaf $\mathcal{O}_X(D)$ by the formula

$$\mathcal{O}_X(D) = i_* \mathcal{O}_U(D|_U),$$

where $U \subset X$ is an open set whose complement is of codimension at least 2 on which D is Cartier and $i \colon U \to X$ is the inclusion. The divisor D is Cartier if $\mathcal{O}_X(D)$ is invertible. We say that D is $\mathbb{Q}$-*Cartier* if some multiple of D is Cartier.

Now we are ready to look at the example below, which is a contraction that results in a singular variety. Understanding divisors on this example is crucial:

Example 2.1. Consider an abelian threefold A. It comes equipped with an involution i. This gives a quotient map $\pi : A \to A/i$ into the Kummer variety of A. There are 64 2-torsion points of A, which induce cyclic quotient singularities of A/i, étale locally isomorphic to $\mathbb{K}^3/i$ given by $i(x, y, z) = (-x, -y, -z)$.

Now consider $f : X \to Y := A/i$ the minimal resolution of A/i. This is simply the blowup of the 64 singular points by [34]. There are 64 exceptional divisors E_i's of f, each of which is isomorphic to $\mathbb{P}^2$ with $\mathcal{O}_{E_i}(E_i) \cong \mathcal{O}_{\mathbb{P}^2}(-2)$. Therefore, for each line class l_i in E_i, $K_X \cdot l = -1$, implying that f is just a contraction of 64 disjoint irreducible rational curve classes that have negative intersection with K_X. This shows that a contraction of K_X-negative curves can result in a singular variety Y.

The cyclic quotient singularity in Example 2.1 has the feature that all even multiples of Weil divisors are Cartier. This is particularly nice, as the canonical divisor K_Y is always locally a rational multiple of a Cartier divisor. This makes the intersection of curves with K_Y into a well-defined operation with rational numbers as the output.

Definition 2.2. A normal variety X is $\mathbb{Q}$-*factorial* if every Weil divisor D of X is $\mathbb{Q}$-Cartier, i.e. there is a large positive integer N such that ND is Cartier on X. X is *factorial* if every Weil divisor is Cartier.

Now consider the resolution map f from Example 2.1. Since Y is $\mathbb{Q}$-factorial, K_Y is $\mathbb{Q}$-Cartier, so that K_Y can be pulled back by f. It turns out that

$$f^*K_Y = K_X - \sum_i \frac{1}{2}E_i.$$

In fact, the coefficients of E_i being negative can be seen by noting that $l_i \cdot f^*K_Y = 0$ and $l_i \cdot K_X < 0$. This suggests that the singularities arising from contractions on smooth varieties should be characterized by the coefficients of the exceptional divisors of the contraction.

To expand the idea of reading off coefficients of exceptional divisors, it is natural to generalize a normal variety Y into a pair (Y, D) of a normal variety Y (not necessarily $\mathbb{Q}$-factorial) with an effective $\mathbb{Q}$-divisor D, i.e. D can be written as

$$D = \sum_i a_i D_i$$

where $a_i \in \mathbb{Q} \cap (0, 1]$ and D_i's are reduced irreducible divisors of Y. We replace the role of the canonical divisor K_Y by the *log canonical divisor* $K_Y + D$. In this case, a birational morphism $f : X \to Y$ is a *log resolution* of (Y, D) if X is nonsingular, $\mathrm{supp}(f^*D)$ is nonsingular, and $\mathrm{supp}(f^*D) \cup \mathrm{exc}(f)$ only admits simple normal crossing (snc for short) singularities. A log resolution exists by [25]. Sometimes, we instead say that the pair $(X, f_*^{-1}D)$ is a log resolution of (Y, D) via f, where $f_*^{-1}D = \sum_i a_i f_*^{-1}D_i$ is the proper

transform of D on X and $f_*^{-1}D_i$ is understood as the proper transform of D_i on X. Now, we are ready to characterize singularities of the pair (Y, D) via a log resolution f:

Definition 2.3. Suppose that (Y, D) is as above such that $K_Y + D$ is $\mathbb{Q}$-Cartier. Fix a log resolution $f : X \to Y$. Denote by $f_*^{-1}D$ the proper transform of D on X (i.e. $f_*^{-1}D = \sum_i a_i f_*^{-1}D_i$). In this case,

$$f^*(K_Y + D) = K_X + f_*^{-1}D + E, \quad E = \sum_i b_i E_i$$

where each E_i is an exceptional divisor of f and $a_i \in \mathbb{Q}$. Then, (Y, D) is

   (1) *terminal* if $E < 0$, i.e. all $b_i < 0$.

   (2) *canonical* if $E \leq 0$.

   (3) *Kawamata log terminal* (klt). if each $b_i < 1$ and $a_i < 1$

   (4) *(purely) log terminal* ((p)lt) if each $b_i < 1$ (this is sometimes written as $\lfloor E \rfloor \leq 0$).

   (5) *log canonical* (lc) if each $b_i \leq 1$.

By [30, Corollary 2.32], the above definition is independent of the choice of log resolution. Moreover, note that if $D \neq 0$, then the log resolution $(X, f_*^{-1}D)$ has canonical singularities (which can be seen by taking further blowups on $(X, f_*^{-1}D)$ along various centers). Note that any smooth variety X is also terminal by the definition.

Remark 2.4. In Definition 2.3, $-b_i$ is called the *discrepancy* of E_i with respect to (X, D). It is written as $a(E_i, X, D)$ in [30, §2.3]. Loc. cit. shows that each discrepancy is independent of the choice of f (where each E_i is interpreted as a discrete valuation on $K(X)$ the field of rational functions on X).

Example 2.5. Suppose that $D = 0$. Then any canonical surface singularity is Gorenstein (i.e. the canonical divisor is Cartier) and is étale locally defined by one of the following equations on $\mathbb{K}^3$ (see [30, Theorem 4.20] for the proof):

   • type A_n: $x^2 + y^2 + z^{n+1} = 0$,

- type D_n: $x^2 + y^2z + z^{n-1} = 0$,

- type E_6: $x^2 + y^3 + z^4 = 0$,

- type E_7: $x^2 + y^3 + yz^3 = 0$,

- type E_8: $x^2 + y^3 + z^5 = 0$,

Example 2.6. For an isolated singularity of a variety X at a point p, the notion of klt and lt coincide. If X has a klt singularity at p, then by [30, Corollary 5.21], it is étale locally a quotient of a canonical singularity $(Y, 0)$ by a cyclic finite group μ_r. If moreover $(Y, 0)$ is nonsingular, then (X, p) is called a *quotient singularity*. In this case, (X, p) is $\mathbb{Q}$-factorial by [30, Proposition 5.15].

When (X, p) is a quotient singularity, we tend to denote it as $\mathbb{K}^d/\mu_r$, where d is the dimension of X and r is the order of the finite cyclic group acting on $\mathbb{K}^d$. If a μ_r-action is of the form $\zeta \cdot (x_1, \dots, x_d) = (\zeta^{a_1} x_1, \dots, \zeta^{a_d} x_d)$ where ζ is a primitive r^{th} root of unity, then denote $\frac{1}{r}(a_1, \dots, a_d)$ to be the singularity $\mathbb{K}^d/\mu_r$ with this μ_r-action.

Consider a pair (X, D) where X is a normal variety and D is an effective divisor of X. Then for which values of $w \in [0, 1]$ is (X, wD) log canonical? By [30, Corollary 2.35], if (X, wD) is log canonical for some w, then so is $(X, w'D)$ for $w' < w$. Thus, we want to find the supremum of all such w. Call this number the *log canonical threshold* (lct) of D. This is a useful quantity, as it encodes how singular the divisor D is when X is smooth near D.

Example 2.7. Suppose that $X \cong \mathbb{K}^2$ and D is a reduced divisor given by an equation $x^a + y^b = 0$. Then, the lct of (X, D) is $\frac{1}{a} + \frac{1}{b}$. In particular, lct of an A_n curve singularity (where D is defined by $x^2 + y^{n+1} = 0$) is $\frac{1}{2} + \frac{1}{n}$.

In the proof of Proposition 3.11, we use one more class of singularities:

Definition 2.8. Suppose that a pair (X, D) is given with D written as a sum of irreducible divisors $\sum_i a_i D_i$ such that $a_i \in [0, 1]$. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Then (X, D) is *divisorially log terminal* iff there is a closed subset $Z \subset X$ such that

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(1) $X \setminus Z$ is smooth and $D|_{X \setminus Z}$ is an snc divisor, and

(2) if $f : Y \to X$ is birational and $E \subset Y$ is an irreducible divisor such that $f(E) \subset Z$ then $a_E < 1$, where a_E is the coefficient of E in the expression

$$f^*(K_X + D) = K_Y + \sum_F a_F F$$

Observe that dlt sits in between lt and lc, in the sense that lt implies dlt and dlt implies lc, but lc does not imply dlt.

2.2. **Main theorems of the MMP.** With the various definitions of singularities of pairs in § 2.1, we would like to understand which curve to contract and see what singularity arises as a result of such contraction. To understand which curves to contract on a given projective variety X, recall that the goal is to intersect curves with a *log canonical divisor* $K_X + D$ associated to a pair (X, D). Thus, we need to understand curves on X as duals to divisors with respect to the intersection pairing on X.

As a generalization of finding a minimal model of a variety, consider a relative setting, where we fix a projective morphism $f : X \to Y$ between normal varieties. The naive goal of (relative log) MMP is to turn $f : X \to Y$ into another map $f' : X' \to Y$ by a sequence of contractions of curves on X that map to points under the map f, so that $K_{X'} + D'$ is f'-nef. In this case, we call X'/Y to be the *relative log minimal model* of $(X, D)/Y$. To carry this out, we need to find a curve E in X that maps to a point via f, with the additional condition that it has a negative intersection with the log canonical divisor $K_X + D$ coming from the pair (X, D). This leads to the following definition:

Definition 2.9. Suppose that $f : X \to Y$ is a projective morphism. Then, a *relative 1-cycle* is a $\mathbb{R}$-formal linear combination $C = \sum_i a_i C_i$ of irreducible, reduced and proper curves C_i such that $f(C_i)$ are points. Two 1-cycles C, C' are called *numerically equivalent* if $C \cdot D = C' \cdot D$ for all Cartier divisor D. The $\mathbb{R}$-linear space of relative 1-cycles modulo numerical equivalence is called $N_1(X/Y)$. The numerically effective cone $\overline{NE}(X/Y)$

is the closure of the subset of $N_1(X/Y)$ generated by effective relative 1-cycles (i.e. $C = \sum a_i C_i$ with $a_i \geq 0$).

Note that $\overline{NE}(X/Y)$ is a convex cone, as any weighted sum of two effective relative 1-cycles is again an effective 1-cycle.

Remark 2.10. If f is a projective morphism to a point (Y is a point), then we write X instead of X/Y in Definition 2.9. Moreover, we omit the word "relative" in the definitions.

Recall that we are interested in obtaining morphisms that contract curves having negative intersection with the log canonical divisor $K_X + D$. To do so, the strategy is to find an f-nef line bundle L that contracts a curve E having a negative intersection with $K_X + D$. If there is a surjection $\mathcal{E} \twoheadrightarrow f_* L$ such that the linear series $\mathcal{E}_y \to H^0(L_y)$ on each fiber $X_y := f^{-1}(y)$ of f has no base points, then we can construct the Y-morphism $\phi_L : X \to \mathbb{P}\mathcal{E}^\vee$ that contracts L-trivial curves.

Since any line bundle is an equivalence class of Cartier divisors (of sections), L corresponds to a hyperplane $L^\perp := \{C \in \overline{NE}(X/Y) \ : \ C \cdot L = 0\}$. Defining $L_{\geq 0} := \{C \in \overline{NE}(X/Y) \ : \ C \cdot L \geq 0\}$, note that $\overline{NE}(X/Y)$ is contained in $L_{\geq 0}$ and $\overline{NE}(X/Y) \cap L^\perp$ is the subcone generated by curves in X that get contracted under the map ϕ_L.

If $\overline{NE}(X/Y) \cap L^\perp$ is of positive dimension and $(K_X + D)$ is not f-nef, then we can replace L (as a $\mathbb{Q}$-Cartier divisor) by $M + t(K_X + D)$ for some $t \in \mathbb{R}_{\geq 0}$ where M is an ample $\mathbb{Q}$-Cartier divisor sufficiently nearby L in $N^1(X)$ (defined similarly to $N_1(X)$) to reduce the dimension of intersection to one. This preserves f-nefness of L. When a hyperplane $(L^\perp)$ only picks up a single ray of a convex cone, it is called an *extremal ray*. If this ray happens to be generated by a class $[C']$ of an irreducible projective curve C, then C is called an *extremal curve*. Observe that if $\overline{NE}(X/Y) \cap L^\perp$ is an extremal ray, generated by one class $[C]$, then we expect ϕ_{mL} to be a morphism for some $m \gg 0$ such that it only contracts curves in fibers of f that are numerically equivalent to $\mathbb{R}_{\geq 0}$-multiples of

$[C]$. This is a desired contraction to perform, in order to make the log canonical divisor $K_X + D$ closer towards being f-nef.

Now we have reduced study of the whole cone $\overline{NE}(X/Y)$ into finding extremal rays $R \subset \overline{NE}(X/Y)$ such that for any $C \in R \setminus 0$, $C \cdot (K_X + D) < 0$. This leads to obtaining the structure of $\overline{NE}(X/Y)$. To state the theorem in this regard, we need one more definition. Consider the subcone $\overline{NE}(X/Y)_{K_X+D\geq0} := \overline{NE}(X/Y) \cap (K_X + D)_{\geq0}$ where $(K_X + D)_{\geq0} := \{C \in N_1(X) : C \cdot (K_X + D) \geq 0\}$. When $K_X + D$ is replaced by a $\mathbb{Q}$-Cartier divisor M, define $\overline{NE}(X/Y)_{M\geq0}$ analogously. The structure of $\overline{NE}(X/Y)$ with respect to $K_X + D$ is given by the Cone Theorem below, from [30, Theorem 3.25(1)(2)]:

Theorem 2.11 (Cone Theorem)**.** *Let (X, D) be a klt pair, D an effective $\mathbb{Q}$-divisor, and $f : X \to Y$ a projective morphism. Then*

(1) There are countably many rational curves $C_j \subset X$ such that $f(C_j) = point$, $0 < -(K_X + D) \cdot C_j \leq 2 \dim X$, and

$$\overline{NE}(X/Y) = \overline{NE}(X/Y)_{K_X+D\geq0} + \sum \mathbb{R}_{\geq0}[C_j]$$

(2) For any $\epsilon > 0$ and f-ample divisor H,

$$\overline{NE}(X/Y) = \overline{NE}(X/Y)_{K_X+D+\epsilon H\geq0} + \sum_{finite} \mathbb{R}_{\geq0}[C_j]$$

This in particular implies that for each $(K_X + D)$-negative extremal curve C_j (which is rational) with $f(C_j) = point$, there is an f-nef line bundle L_j such that $\overline{NE}(X/Y) \cap L_j^\perp = \mathbb{R}_{\geq0}[C_j]$. The corresponding rational map ϕ_L given by the (relative) complete linear series is not necessarily basepoint free. We can draw similar conclusion for any $(K_X + D)$-negative *extremal face*, a face of $\overline{NE}(X/Y)$ generated by a finite number of such C_j's. The Contraction Theorem below gives the behavior of such rational maps ϕ_L:

Theorem 2.12 (Contraction Theorem)**.** *Take the same assumptions as in Theorem 2.11. Then,*

(1) *Let $F \subset \overline{NE}(X/Y)$ be a $(K_X + D)$-negative extremal face. Then there is a unique Y-projective morphism* $\mathrm{cont}_F : X \to Z$, *called the* contraction of F, *such that* $(\mathrm{cont}_F)_* \mathcal{O}_X = \mathcal{O}_Z$ *and an irreducible curve $C \subset X$ is mapped to a point by* cont_F *iff $[C] \in F$.*

(2) *Let F and* $\mathrm{cont}_F : X \to Z$ *as above. If L is a line bundle on X such that $L \cdot C = 0$ for every curve $[C] \in F$, then there is a line bundle L_Z on Z with $L \cong \mathrm{cont}_F^* L_Z$.*

So far, we only intersect $K_X + D$ with curves in X with zero-dimensional image under f. Thus, both $K_X + D$ and $K_{X/Y} + D$ have the exact same properties as a dual function on $N_1(X/Y)$. Therefore, we freely interchange between $K_X + D$ and $K_{X/Y} + D$ for the (relative log) MMP, and call both the (relative) log canonical divisor.

In the case of a nonsingular projective threefold X with a map f to a point, Mori in [35, Theorem 3.3 and Corollary 3.4] lists the structure of the contraction of a K_X-negative extremal ray R, which is summarized below in Theorem 2.13. To understand the statement, we need the following definition: an extremal ray $R = \mathbb{R}_{\geq 0}[C] \subset N_1(X)$ is *numerically effective* if $C \cdot M \geq 0$ for any effective divisor D.

Theorem 2.13. *Let X be a nonsingular projective threefold. If a K_X-negative extremal ray R is not numerically effective, then* $\mathrm{cont}_F : X \to Z$ *is birational and the exceptional locus* $\mathrm{exc}(\mathrm{cont}_R)$ *is an irreducible divisor E with* $\dim(\mathrm{cont}_R(E)) \leq 1$. *There are five cases on the structure of* cont_F *near E:*

(1) $\mathrm{cont}_R(E)$ *is a nonsingular curve and Z is nonsingular.* $\mathrm{cont}_F|_E : E \to \mathrm{cont}_R(E)$ *is a $\mathbb{P}^1$-bundle.*

(2) $q = \mathrm{cont}_R(E)$ *is a point and Z is nonsingular. $E \cong \mathbb{P}^2$ and $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$.*

(3) $q = \mathrm{cont}_R(E)$ *is a point, $E \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$. Z has a factorial singularity at q given by*

$$\mathbb{K}[\![x, y, z, w]\!]/(x^2 + y^2 + z^2 + w^2)$$

which is an ordinary double point 3-fold singularity.

(4) $q = \mathrm{cont}_R(E)$ *is a point, E is isomorphic to a singular quadric surface Q in $\mathbb{P}^3$, and* $\mathcal{O}_E(E) \cong \mathcal{O}_Q(-1)$. *Z has a factorial singularity at q given by*

$$\mathbb{K}[\![x, y, z, w]\!]/(x^2 + y^2 + z^2 + w^3)$$

(5) $q = \mathrm{cont}_R(E)$ *is a point, $E \cong \mathbb{P}^2$, and $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^2}(-2)$. Z has a $\mathbb{Q}$-factorial singularity at q given by*

$$\mathbb{A}^3_{x,y,z}/i, \quad i(x, y, z) = (-x, -y, -z)$$

As a generalization of the nice situation in Theorem 2.13, consider a projective morphism $f : X \to Y$ with $K_X + D$ $\mathbb{Q}$-Cartier but not f-nef, and $\mathrm{cont}_R : X \to Z$ for some extremal ray R is a birational morphism that contracts an irreducible divisor. We call such contractions a *divisorial contraction*. By [30, Proposition 3.36], any divisorial contraction on a dlt pair (X, D) preserves $\mathbb{Q}$-Cartierness of log canonical divisors, i.e. the log canonical divisor $K_Z + D_Z := (\mathrm{cont}_R)_*(K_X + D)$ on the image (Z, D_Z) is $\mathbb{Q}$-Cartier. Moreover, discrepancies on (Z, D_Z) are not smaller than that on (X, D) by [30, Lemma 3.38], so that by Definition 2.3 and Remark 2.4, a divisorial contraction preserves singularity types of pairs, possibly except for the case of lc singularity.

However, not all contractions cont_R are divisorial. What happens when cont_R is a *small contraction*, i.e. birational but the codimension of $\mathrm{exc}(\mathrm{cont}_R)$ is bigger than 1? In this case, $\mathbb{Q}$-Cartierness of the log canonical divisor is not preserved, as seen in [30, Case 3 of 2.6]. This is particularly bad, as the singularity we obtain is much worse than Definition 2.3. This is the case where taking a contraction is not desired. Instead, we take a birational map called *flip* on (X, D) which is an isomorphism on $X \setminus \mathrm{exc}(\mathrm{cont}_R)$. We restate the definition given in [30, Definition 3.33] in the modern framework:

Definition 2.14. Given a pair (X, D) with X normal and $K_X + D$ $\mathbb{Q}$-Cartier, suppose that we have a small contraction $g = \mathrm{cont}_R : (X, D) \to (Z, D_Z)$ of a $K_X + D$-negative

extremal ray R. Then, the *flip* of g is a rational map $h : (X, D) \dashrightarrow (X', D')$ where D' is the birational transform of D and $K_{X'} + D'$ is $\mathbb{Q}$-Cartier which fits into the following commutative diagram

$$
\begin{array}{ccc}
(X, D) & \overset{h}{\dashrightarrow} & (X', D') \\
& \searrow{\scriptstyle g} \quad \swarrow{\scriptstyle g'} & \\
& (Z, D_Z) &
\end{array}
$$

such that

(1) g' is proper birational with $\mathrm{codim}(\mathrm{exc}(g')) \geq 2$, and

(2) $K_{X'} + D'$ is g'-ample.

Often, we will say a flip h instead of specifying the small contraction g.

Remark 2.15. Starting with a small contraction $g : (X, D) \to (Z, D_Z)$ from a klt pair (X, D), the flip of g exists by [8, Corollary 1.4.1], and is unique by [30, Corollary 6.4].

The flip corresponding to an extremal ray R is indeed analogous to a divisorial contraction, except that the Picard rank does not drop by 1 (it stays the same). First, $\mathbb{Q}$-Cartierness of log canonical divisors is preserved under a flip. Second, a flip preserves the singularity types by [30, Lemma 3.38]. Third, if (X, D) is a projective $\mathbb{Q}$-factorial dlt pair, then X' is also $\mathbb{Q}$-factorial. Finally, a flip turns the $K_X + D$-negative extremal curve C (that generates R) into a curve $C' \subset \mathrm{exc}(g')$, and $(K_{X'} + D') \cdot C' > 0$. Hence, we can think of a flip as a step towards turning $K_X + D$ into a nef divisor.

Remark 2.16. In general, the flip of g is difficult to describe explicitly. Even when $D = 0$, the flip of g is characterized by

$$
g' : X' = \mathrm{Proj}_Z \bigoplus_{m=0}^{\infty} g_* \mathcal{O}_X(mK_X) \to Z
$$

but it seems hopeless in general to compute various pushforwards on a singular variety Z. Instead, we adapt to an idea coming from the Weak Factorization Theorem [3, Theorem

0.3.1], which roughly states that a birational map between proper nonsingular varieties can be decomposed into a sequence of blowups and blowdowns. Motivated by this idea, we construct two examples of flips explicitly in Section 5 by constructing explicit sequences of blowups and blowdowns. Those flips will be used in Section 6.

Outcome of the MMP. Now we are ready to precisely state the goal of the (relative log) *minimal model program*. Start with a klt pair (X, D) and a projective morphism $f : X \to Y$ between normal varieties over an algebraically closed field $\mathbb{K}$ of characteristic zero. The goal is to find a Y-birational model (X', D') of (X, D) such that one of the following two is true:

- *Mori fiber space*: $\text{cont}_R : X' \to B$, where R is a $K_{X'} + D'$-negative extremal ray, is a projective morphism over B of smaller dimension than X, and the Picard rank $\rho(B) = \rho(X) - 1$.

- *Minimal model*: $K_{X'} + D'$ is nef over Y (so that X and X' are in fact Y-birational).

Mori fiber spaces are also called *log Fano fibrations*, because the Picard rank of each fiber is 1 and $K_X + D$ is antiample on each fiber. $\mathbb{P}^1$-bundles are examples of Mori fiber spaces. It is easy to see that there are no minimal models for such bundles. We will see below that both minimal models and Mori fiber spaces should cover all cases of interest.

Steps of the MMP. The MMP is a recursive procedure that turns (X, D) into a birational model, which is either a minimal model or a Mori fiber space. Fix the base B. The following procedure is obtained by a sequence of birational maps discussed above:

Step 1: Start with a klt pair (X, D) and a projective morphism $f : (X, D) \to Y$.

Step 2: Is $K_X + D$ f-nef?

- If yes, $(X, D)/Y$ is a minimal model. End.

- Otherwise, choose a $(K_X + D)$-negative extremal ray $R \subset \overline{NE}(X/Y)$ by Theorem 2.11.

Step 3: Consider the Y-morphism $\text{cont}_R : (X, D) \to (Z, D_Z)$ (exists by Theorem 2.12). Is $\dim(Z)$ smaller than $\dim(X)$?

- If yes, $\text{cont}_R : (X, D) \to Z$ is a Mori fiber space. End.
- Otherwise, move to the next step.

Step 4: Is the codimension of $\text{exc}(\text{cont}_R)$ bigger than 1?

- If yes, take a flip $h : (X, D) \dashrightarrow (X', D')$ (this can be done by Remark 2.15) and go back to Step 1 with $(X', D')/Y$ instead of $(X, D)/Y$.
- Otherwise, go back to Step 1 with $(Z, D_Z)/Y$ instead of $(X, D)/Y$.

It is not clear from the algorithm of MMP whether it terminates or not. By the celebrated result in [8, Theorem 1.2], usually annotated as BCHM, MMP terminates for some choices of extremal rays when either D is f-big and $K_X + D$ is f-pseudoeffective or $K_X + D$ is f-big. Since we will mainly use the bigness, we recall the definition of f-big below and direct readers to [10, §1.6.3] for the definition of f-pseudoeffective:

Definition 2.17. Given a irreducible projective variety X, a $\mathbb{Q}$-Cartier divisor D is *big* if $[D] = [A] + [N]$, where A is an ample $\mathbb{Q}$-divisor and N is an effective $\mathbb{Q}$-divisor. In the relative setting, where $f : X \to Y$ is projective with Y an irreducible variety, then a $\mathbb{Q}$-Cartier divisor D on X is f-big if D_η is big on X_η, where $\eta \in S$ is the generic point.

Note that bigness is a mild generalization of ampleness in $N^1(X)$. Moreover, f-pseudoeffectivity of D roughly states that on some fiber of f, D is realized as a limit of effective $\mathbb{R}$-divisors in $N^1(X/B)$. Now we are ready to consider an example where the conditions of [8, Theorem 1.2] hold.

Example 2.18. Consider a smooth projective morphism $\pi : X \to \Delta$ of fiber dimension 2 into a DVR Δ with $K_{X/\Delta}$ antiample. Call η the generic point of Δ. Suppose that we are given a general pair (X, D) with D/S flat, reduced, and proper of fiber dimension 1 such that D_η is smooth, $[D] \in \mathbb{R}_{\geq 0}[-K_X] \subset N^1(X/\Delta)$, and $K_X + wD$ is π-nef for some $w \in \mathbb{Q} \cap (0, 1)$. In this case, D is ample and $K_X + wD$ is π-pseudoeffective. In general,

the pair (X, D) can have very nasty singularities, so we cannot use MMP directly on $(X, D)/\Delta$.

Now consider a log resolution $f : (\tilde{X}, \tilde{D}) \to (X, D)$. Notice that f only modifies the central fiber. Hence, $(\tilde{D}_\eta \hookrightarrow \tilde{X}_\eta) \cong (D_\eta \hookrightarrow X_\eta)$, so that $\tilde{D}$ is $(f \circ \pi)$-big on $\tilde{X}$. Since $(K_{\tilde{X}} + w\tilde{D})_\eta = (K_X + D)_\eta$ is pseudoeffective, $K_{\tilde{X}} + w\tilde{D}$ remains $(f \circ \pi)$-pseudoeffective. Hence, the conditions of [8, Theorem 1.2] are satisfied on the log resolution.

Moreover, if $K_X + wD$ is π-ample, then $K_X + wD$ is π-big. A similar argument as above shows that $K_{\tilde{X}} + w\tilde{D}$ remains π-big.

This allows us to use MMP on a 1-parameter family of surfaces considered in Section 3.

Remark 2.19. Note that finding a minimal model involves choosing a sequence of extremal rays, so that a minimal model is not unique. In fact, given a klt pair (X, D), minimal models are birational by a sequence of *flops*. Since we do not use *flops* in subsequent sections, we do not define it in this paper. Instead, read [30, Definition 6.10] for self-interest.

2.3. **The log canonical program.** Suppose that given a lc pair (X, D) with $K_X + D$ f-nef where $f : X \to Y$ is a projective morphism, we would like to find a birational model $(\overline{X}, \overline{D})/Y$ of $(X, D)/Y$ where $K_{\overline{X}} + \overline{D}$ is ample. This is particularly useful for studying moduli spaces of varieties, as the ampleness of the log canonical divisor implies that the pair $(\overline{X}, \overline{D})/Y$ has only finitely many automorphisms over Y. We call such pair $(\overline{X}, \overline{D})/Y$ a *(log) canonical model* of $(X, D)/Y$, if X is still B-birational to $\overline{X}$.

To obtain a canonical model from a minimal model $(X, D)/Y$, all we need to do is to contract all curves C with zero-dimensional images in Y such that $C \cdot (K_X + D) = 0$. If $(\overline{X}, \overline{D})/B$ is the result of such contraction (if such contraction exists), then $K_{\overline{X}} + \overline{D}$ is ample by Kleiman's criterion [30, Theorem 1.18] if $K_{\overline{X}} + \overline{D}$ is $\mathbb{Q}$-Cartier. Since $K_X + D$ is already nef, such contraction exists if a linear series from $m(K_X + D)$ is basepoint free

for m sufficiently high. This is guaranteed when X is a threefold by [28, Theorem 1.1], which we state it below:

Theorem 2.20. *(log Abundance Theorem for threefolds) If X is a threefold, (X, D) is lc, and $K_X + D$ is nef, then $|m(K_X + D)|$ is basepoint free for some m.*

This implies the relative version, as above theorem has no properness assumption on X and by restricting Y to an affine open neighborhood, the conditions of above is met. Hence, we can obtain a log canonical model $(\overline{X}, \overline{D})/Y$ of a minimal model (X, D). Moreover, it is easy to see that $(\overline{X}, \overline{D})$ has lc singularities. Instead of trying to find such m to construct $(\overline{X}, \overline{D})/Y$, [30, Theorem 3.52] gives a construction of $(\overline{X}, \overline{D})/Y$ without finding this m:

$$(2.1) \qquad \overline{X} = \bigoplus_{m \geq 0} f_* \mathcal{O}_X(mK_X + \lfloor mD \rfloor)$$

Loc. cit. also shows that the log canonical model is unique as well. In fact, Loc. cit. shows that given any log canonical pair $(X, D)/Y$ with X projective over Y (without the assumption that $K_X + D$ is Y-nef), if a log canonical model exists, then it is unique and is given by above. This implies that the log canonical model $(\overline{X}, \overline{D})/Y$ is independent of the minimal model of $(X, D)/Y$ that we find.

Henceforth, given any klt pair (X, D) where X is a threefold and is projective over Y, we can combine the MMP and (2.1) to obtain the *(log) canonical program*:

Step 1: Start with a klt pair (X, D) and a projective morphism $f : (X, D) \to Y$, where X is a threefold. Assume that either D is f-big and $K_X + D$ is f-pseudoeffective or $K_X + D$ is f-big.

Step 2: Run MMP on $(X, D)/Y$ (by [8, Theorem 1.2]) to get a minimal model $(X', D')/Y$. Is $K_{X'} + D'$ nef over B?

  • If yes, $(X', D')/Y$ is a minimal model. Move to the next step

- Otherwise, $(X', D')/Y$ is a Mori fiber space. End, as (X, D) does not admit a log canonical model.

Step 3: Use the log-pluricanonical series coming from multiples of $K_{X'} + D'$ to construct log canonical contraction map $(X', D') \to (X'', D'')$ (works by Theorem 2.20).

Step 4: Is the fiber dimension of X'' the same as X?

- If yes, (X'', D'') is the log canonical model. End.

- Otherwise, $\dim X'' < \dim X$, and $(X, D)/Y$ does not admit the log canonical model. End.

3. Moduli spaces of 'almost K3' stable log surfaces

In this section, we collect fundamental results on moduli of stable log surfaces of a particular kind that are used throughout this paper. These log surfaces consist of a rational surface and a divisor whose class is proportional to the canonical class, and which is taken with a weight such that the log canonical divisor is just barely ample (hence the name 'almost K3'). Hacking pioneered the study of such surfaces in [19] and [20], hence those surfaces are called Hacking stable in other places in the literature [18, 14]. Our treatment closely follows his work and benefits greatly from the subsequent enhancements due to Abramovich and Hassett [2].

All objects are over an algebraically closed field $\mathbb{K}$ of characteristic 0. We fix a pair of relatively prime positive integers (m, n) with $m \leq n$. After the general foundations in § 3.1–§ 3.4, we take $(m, n) = (2, 3)$.

3.1. **Stable log surfaces.** The following definition is motivated by [20], where a similar object in the context of plane curves is called a stable pair.

Definition 3.1 (Stable log surface). An *almost K3 semi-stable log surface* over $\mathbb{K}$ is a pair (S, D) where S is a projective, reduced, connected, Cohen-Macaulay surface over $\mathbb{K}$ and D is an effective Weil divisor on S such that

    (1) no component of D is contained in the singular locus of S;

    (2) the pair $(S, m/n \cdot D)$ is semi log canonical;

    (3) the divisor class $nK_S + mD$ is linearly equivalent to zero;

    (4) we have $\chi(\mathcal{O}_S) = 1$.

An *almost K3 stable log surface* is an almost K3 semi-stable log surface (S, D) such that for some $\epsilon > 0$

    (1) the pair $(S, (m/n + \epsilon) \cdot D)$ is semi log canonical (slc for short);

    (2) $K_S + (m/n + \epsilon) \cdot D$ is ample.

For brevity, from now on we refer to an almost K3 stable log surface simply as a *stable log surface*. This is also called We also suppress the choice of (m, n), which remains fixed throughout this section, and equal to $(2, 3)$ after § 3.4.

Remark 3.2. If S is smooth, then S is a del Pezzo surface. The case of $S \cong \mathbb{P}^2$ (and its degenerations) was studied by Hacking in [19] and [20]. Our interest in this paper is the case of $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ (and its degenerations) and $(m, n) = (2, 3)$.

We recall some terms in the definition above, mainly to set the conventions. We assume throughout that our Weil divisors are Cartier in codimension 1. That is, there exists an open subset $U \subset S$ whose complement is of codimension at least 2 such that the restriction of the divisor to U is Cartier. In Definition 3.1, this is guaranteed by the first requirement.

A coherent sheaf F on S is *divisorial* if there exists an open inclusion $i \colon U \to S$ with complement of codimension at least 2 such that $i^* F$ is invertible and

$$F = i_*(i^* F),$$

A divisorial sheaf is isomorphic to $\mathcal{O}_S(D)$ for some Weil divisor D on S. Indeed, if $i^* F \cong \mathcal{O}_U(D^\circ)$, where D° is a Cartier divisor on U, then we may take $D = \overline{D^\circ}$. Two Weil divisors D_1 and D_2 are linearly equivalent if and only if the sheaves $\mathcal{O}_S(D_1)$ and $\mathcal{O}_S(D_2)$ are isomorphic. The divisor class K_S is the linear equivalence class corresponding to the divisorial sheaf ω_S. For a divisorial sheaf F and $n \in \mathbb{Z}$, denote by $F^{[n]}$ the divisorial sheaf $i_*(i^* F^{\otimes n})$. This operation corresponds to multiplication by n on the associated divisors.

The semi log canonical condition in Definition 3.1 entails the following:

(1) S has at worst normal crossings singularities in codimension 1.

(2) Let K_S be the Weil divisor associated to the dualizing sheaf ω_S. Then the $\mathbb{Q}$-Weil divisor $K_S + (m/n + \epsilon) \cdot D$ is $\mathbb{Q}$-Cartier.

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(3) Let $S^\nu \to S$ be the normalization, $B \subset S^\nu$ the pre-image of the double curve (the divisor defined by the different ideal), and $D^\nu \subset S^\nu$ the pre-image of D. Then the pair $(S^\nu, (m/n + \epsilon) \cdot D^\nu + B)$ is lc (see Definition 2.3).

Since $nK_S + m \cdot D$ is linearly equivalent to 0, if $K_S + (m/n + \epsilon) \cdot D$ is $\mathbb{Q}$-Cartier, then both K_S and D are $\mathbb{Q}$-Cartier. Note that if (S, D) satisfies the conditions of Definition 3.1 for a particular ϵ, then it also satisfies the definitions for all $\epsilon' < \epsilon$.

3.2. **Families of stable log surfaces.** Having defined stable log surfaces, we turn to families of them. Ideally, the passage from objects to families ought to be straightforward. A family of stable log surfaces should be a flat morphism whose fibers are stable log surfaces. However, this turns out to be too naïve. To ensure a well-behaved moduli space—one in which numerical invariants are locally constant—additional conditions are needed. There are subtleties in choosing the right choice of conditions for families of log varieties in general. For our case, however, there is a clear answer, developed in [19], which we follow.

Let B be a $\mathbb{K}$-scheme, and $\pi \colon S \to B$ a flat, Cohen-Macaulay morphism of relative dimension 2 with geometrically reduced fibers. An *effective relative Weil divisor* on S is a subscheme $D \subset S$ such that there exists an open subset $U \subset S$ satisfying the following conditions:

(1) for every geometric point $b \to B$, the complement of U_b in S_b is of codimension at least 2;

(2) $D|_U \subset U$ is Cartier (its ideal sheaf is invertible) and flat over B;

(3) D is the scheme-theoretic closure of $D|_U$.

A *relative Weil divisor* is a formal difference of effective relative Weil divisors. A *divisorial sheaf* is a coherent sheaf F on S such that i^*F is locally free and $F = i_*i^*F$, where $i \colon U \to S$ is the inclusion of an open set as above. A relative Weil divisor D gives a divisorial sheaf $\mathcal{O}_S(D)$, and every divisorial sheaf is of this form. Given a divisorial sheaf

F and $n \in \mathbb{Z}$, we have a divisorial sheaf $F^{[n]}$ defined as before. If the geometric fibers S_b are slc, then $\omega_{S/B}$ is a divisorial sheaf [19, Example 8.18].

Let A be a $\mathbb{K}$-scheme with a map $A \to B$. Let $\pi \colon S \to B$ be as before. Let D be a effective relative Weil divisor on S. Set $S_A = S \times_A B$. The *divisorial pullback* of D to A, denoted by $D_{(A)}$, is the divisor given by the closure of $D|_U \times_B A$ in S_A. Note that $D_{(A)}$ may not be equal to the subscheme $D \times_A B$ of S_A. The divisorial pull-back of a non-effective relative divisor is defined by linearity. Likewise, given a divisorial sheaf F on S, its divisorial pull-back $F_{(A)}$ is defined by

$$F_{(A)} = i_{A*} i_A^* F,$$

where $i_A \colon U \times_B A \to S_A$ is the open inclusion pulled back from $U \to S$. Again, the divisorial pull-back $F_{(A)}$ may not be equal to the usual pullback $F_A = F \times_B A$. To compare the two, observe that we always have a map

(3.1) $$F_A \to F_{(A)}.$$

This map is an isomorphism if F_A is divisorial. We say that F *commutes with base change* if for every $\mathbb{K}$-scheme A with a map $A \to B$, the map in (3.1) is an isomorphism, or equivalently, the usual pullback F_A is divisorial. To check that F commutes with base change, it suffices to check that it commutes with the base change for the inclusions of closed points into S [19, Lemma 8.7]. Furthermore, if F commutes with base change, then F is flat over B [19, Lemma 8.6]. Plainly, if F is locally free, then it commutes with base change. Furthermore, by Nakayama's lemma, it is easy to see that if F commutes with base change, and F_b is invertible for all $b \in B$, then F is invertible.

Following [19, Definition 2.14], we make the following definition.

Definition 3.3 ($\mathbb{Q}$-Gorenstein family)**.** Let B be a $\mathbb{K}$-scheme. A $\mathbb{Q}$-*Gorenstein family of log surfaces* over B is a pair $(\pi \colon S \to B, D \subset S)$ where π is a flat Cohen–Macaulay

morphism with geometric fibers of dimension 2 with slc singularities, and $D \subset S$ is a relative effective Weil divisor such that the following hold:

(1) $\omega_\pi^{[i]}$ commutes with base change for every $i \in \mathbb{Z}$, and for every geometric point $b \to B$, there exists an n such that $\omega_{S_b}^{[n]}$ is invertible;

(2) $\mathcal{O}_S(D)^{[i]}$ commutes with base change for every $i \in \mathbb{Z}$.

A $\mathbb{Q}$-Gorenstein family of *stable* log surfaces is a family as above with π proper where all geometric fibers are stable log surfaces.

By [19, Lemma 8.19], if $\mathcal{O}_S(-D)$ commutes with base change, then for every $A \to B$, the divisor D is flat over B and the divisorial pullback $D_{(A)}$ agrees with the usual pullback $D_A = D \times_B A$. In particular, the two possible notions of the fiber of (S, D) over $b \in B$ agree.

3.3. **The canonical covering stack and the index condition.** The analogue of Definition 3.3 without the divisor is called a *Kollár family*. Explicitly, a *Kollár family* of surfaces is a flat, Cohen–Macaulay morphism $\pi \colon S \to B$ with slc fibers satisfying the following conditions

(1) $\omega_\pi^{[i]}$ commutes with arbitrary base change for all $i \in \mathbb{Z}$;

(2) for every geometric point $b \to B$, there exists an n such that $\omega_{S_b}^{[n]}$ is invertible.

Let $\pi \colon S \to B$ be a Kollár family of surfaces. The *canonical covering stack* of S/B is the stack

$$\mathcal{S} = \left[\operatorname{spec} \left(\bigoplus_{n \in \mathbb{Z}} \omega_\pi^{[n]} \right) \big/ \mathbb{G}_m \right],$$

where the $\mathbb{G}_m$ action is given by the grading. By construction, $\mathcal{S} \to B$ is flat and Gorenstein. Furthermore, by [2, Theorem 5.3.6], the natural map $p \colon \mathcal{S} \to S$ is the coarse space map; it is an isomorphism over the locus where ω_π is invertible; and we have $p_* \omega_{\mathcal{S}/B}^n = \omega_{S/B}^{[n]}$. Furthermore, if $U \subset S$ is an open subset such that $\omega_\pi^{[N]}\big|_U$ is invertible,

then we have

$$\mathcal{S} \times_S U \cong \left[\operatorname{spec}\left(\bigoplus_{n=0}^{N-1} \omega_\pi^{[n]}\big|_U \right) \big/ \mu_N \right].$$

Thus, $\mathcal{S}$ is a cyclotomic Deligne–Mumford stack in the language of [2].

The canonical covering stack provides a convenient conceptual and technical framework to deal with the Kollár condition that $\omega_\pi^{[i]}$ commute with base change. It becomes very convenient if it *also* takes care of the second condition in Definition 3.3. This motivates the following discussion.

Let (S, D) be a stable log surface over $\mathbb{K}$. Let $\mathcal{S} \to S$ be the canonical covering stack and $\mathcal{D} \subset \mathcal{S}$ the divisorial pullback of D, namely the divisor obtained by taking the closure of $D|_U \times_S \mathcal{S}$ where $U \subset S$ is an open subset with complement of codimension at least 2 on which D is Cartier.

Definition 3.4 (Index condition). We say that a stable log surface (S, D) satisfies the *index condition* if $\mathcal{D} \subset \mathcal{S}$ is a Cartier divisor.

The reason for the term "index condition" is as follows. Let $s \in S$ be a point. The *index* of S at s is the smallest positive integer N such that $\omega_S^{[N]}$ is invertible at s. Likewise, the *index* of D at s is the smallest positive integer M such that $\mathcal{O}_S(D)^{[M]}$ is invertible at s. The linear equivalence $nK_S + mD \sim 0$ implies that we have an isomorphism

$$\omega_S^{-n} \cong \mathcal{O}_S(D)^{[m]}.$$

The condition in Definition 3.4 holds if and only if $\gcd(m, M) = 1$. Thus, Definition 3.4 is a condition on the index of D.

3.4. **The moduli stack.** Let $\mathfrak{F}$ be the category fibered in groupoids over the category of $\mathbb{K}$-schemes whose objects over B are $\mathbb{Q}$-Gorenstein families of stable log surfaces over B such that all geometric fibers satisfy the index condition. Morphisms in $\mathfrak{F}$ are isomorphisms over B.

Theorem 3.5 (Existence of the moduli stack)**.** $\mathfrak{F}$ *is a Deligne–Mumford stack, locally of finite type over* $\mathbb{K}$.

Thanks to modern technology, it is now possible to give a short proof of this theorem. Much of the heavy lifting is done by [2] and [37]. Before we prove the theorem, we recast $\mathfrak{F}$ in a more amenable form.

Let $\mathfrak{G}$ be the category fibered in groupoids over the category of $\mathbb{K}$-schemes whose objects over B are pairs $(\pi\colon S \to B, \mathcal{D} \subset \mathcal{S})$, where

    (1) π is a flat, proper, Kollár family of surfaces,

    (2) $\mathcal{S} \to S$ is the canonical covering stack,

    (3) $\mathcal{D} \subset \mathcal{S}$ is an effective Cartier divisor flat over B,

such that, for every geometric point $b \to B$, the pair (S, D) is a stable log surface, where D is the coarse space of $\mathcal{D}$.

Proposition 3.6. *The categories* $\mathfrak{G}$ *and* $\mathfrak{F}$ *are equivalent as fibered categories over the category of* $\mathbb{K}$*-schemes.*

Proof. We have a natural transformation $\mathfrak{G} \to \mathfrak{F}$, defined as follows. Consider an object $(\pi\colon S \to B, \mathcal{D} \subset \mathcal{S})$ of $\mathfrak{G}$ over B. Let D be the coarse space of $\mathcal{D}$. Using that $\mathcal{D}$ is a Cartier divisor and that $\mathcal{S} \to S$, we can check that $\mathcal{O}_S(D)^{[n]}$ commutes with base change for all $n \in \mathbb{Z}$ (see [2, Theorem 5.3.6]). Therefore, $(\pi\colon S \to B, D \subset S)$ is an object of $\mathfrak{F}$ over B.

We now show that the transformation $\mathfrak{G} \to \mathfrak{F}$ defined above is an isomorphism. To do so, let us construct an inverse. Let $(\pi\colon S \to B, D \subset S)$ be an object of $\mathfrak{F}$ over B. Let $\mathcal{S} \to S$ be the canonical covering stack, and $\mathcal{D} \subset \mathcal{S}$ the divisorial pullback. Since $D \subset S$ is a $\mathbb{Q}$-Cartier divisor, so is $\mathcal{D} \subset \mathcal{S}$. Furthermore, by the index condition, for every geometric point $b \to B$, the divisor $\mathcal{D}_{(b)}$ is Cartier. By [19, Lemma 8.25], it follows that $\mathcal{D}$ is Cartier. Thus, $(\pi\colon S \to B, \mathcal{D} \subset \mathcal{S})$ is an object of $\mathfrak{G}$ over B. This transformation provides the required inverse. $\qquad\square$

Remark 3.7. Let (S, D) be a stable log surface. Then $-K_S$ is ample, so $h^0(K_S) = h^2(\mathcal{O}_S) = 0$. Since $\chi(\mathcal{O}_S) = 1$ and $h^0(\mathcal{O}_S) = 1$, we also have $h^1(\mathcal{O}_S) = 0$. Thus, $h^i(\mathcal{O}_S) = 0$ for all $i > 0$.

Proof of Theorem 3.5. By Proposition 3.6, we may work with $\mathfrak{G}$ instead of $\mathfrak{F}$. We first show that $\mathfrak{G}$ is an algebraic stack, locally of finite type.

Let Orb^λ be the moduli of polarized orbispaces defined in [2, Section 3] (called Sta^λ in loc. cit.). We have a map $\mathfrak{G} \to \mathrm{Orb}^\lambda$ given by

$$(S \to B, \mathcal{D} \subset \mathcal{S}) \mapsto (\mathcal{S} \to B, \omega_{\mathcal{S} \to B}^{-1}).$$

Since Orb^λ is an algebraic stack locally of finite type, it suffices to show that for every scheme B with a map $\phi: B \to \mathrm{Orb}^\lambda$, the fiber product $\mathfrak{G} \times_\phi B$ is an algebraic stack.

Let B be a scheme with a map $\phi: B \to \mathrm{Orb}^\lambda$ corresponding to a family of polarized orbispaces $(\pi: \mathcal{S} \to B, \lambda)$. After passing to an étale cover, we may assume that the polarization λ comes from a line bundle $\mathcal{L}$ on $\mathcal{S}$. Let $\mathfrak{H} \to B$ be the Hilbert stack of π. This is the stack whose objects over a B-scheme A are substacks $\mathcal{D} \subset \mathcal{S}_A$ flat over A. By [37, Theorem 1.1], $\mathfrak{H} \to B$ is an algebraic space locally of finite type. We show that $\mathfrak{G} \times_\phi B$ is isomorphic to a locally closed substack of $\mathfrak{H}$.

There exists an open substack $U \subset \mathfrak{H}$ with the property that a map $A \to \mathfrak{H}$ given by $(\pi: \mathcal{S}_A \to A, \mathcal{D} \subset \mathcal{S}_A)$ factors through U if and only if

(1) $\mathcal{D} \subset \mathcal{S}_A$ is a Cartier divisor (its ideal sheaf is invertible);

(2) π is Gorenstein;

(3) we have $\chi(\mathcal{O}_{S_a}) = 1$, and for every geometric point $a \to A$, there exists an $\epsilon > 0$ such that $(S_a, (m/n + \epsilon) \cdot D_a)$ is semi log canonical, where (S_a, D_a) is the coarse space of $(\mathcal{S}_a, \mathcal{D}_a)$.

(4) the locus of points in $\mathcal{S}_a$ with non-trivial automorphism groups has codimension at least 2.

The openness of the first condition follows by Nakayama's lemma. See [2, Section 4 and Appendix A] for the openness of the Gorenstein and semi log canonical property. The openness of the last property follows from semi-continuity of fiber dimensions in the inertia stack $I\mathcal{S} \to B$.

There exists a closed substack $V \subset U$ with the property that a map $B \to U$ factors through V if and only if, in addition to the conditions above, we have

(5) for every geometric point $b \to B$, the line bundles $\mathcal{L}_b \otimes \omega_{\mathcal{S}_b}$ and $\mathcal{O}_{\mathcal{S}_b}(\mathcal{D}_b)^m \otimes \omega_{\mathcal{S}_b}^n$ are trivial.

Since $h^0(\mathcal{O}_{\mathcal{S}_b}) = 1$ and $h^i(\mathcal{O}_{\mathcal{S}_b}) = 0$ for all $i > 0$, this condition is equivalent to saying that the line bundles $\mathcal{L} \otimes \omega_\pi$ and $\mathcal{O}_\mathcal{S}(\mathcal{D})^m \otimes \omega_\pi^n$ are pull-backs of line bundles from B. That this is a closed condition follows from [36, III.10].

It is now easy to see that $\mathfrak{G} \times_\phi B$ is isomorphic to V.

Since the automorphism group of a stable pair is finite [26, Theorem 11.12], the stack $\mathfrak{G}$ is Deligne–Mumford. $\square$

It is not clear that $\mathfrak{F}$ is of finite type for two reasons. Firstly, we have not put any numerical conditions on (S, D). Secondly, and more seriously, there is no *a priori* lower bound on the ϵ in Definition 3.1. The problem goes away if we define away these two reasons.

Fix an $\epsilon > 0$ and a positive rational number N. Denote by $\mathfrak{F}_{\epsilon,N}$ the open substack of $\mathfrak{F}$ that parametrizes stable log surfaces that satisfy the definitions of Definition 3.1 with the given ϵ and have $K_S^2 \leq N$.

Proposition 3.8. *$\mathfrak{F}_{\epsilon,N}$ is an open substack of $\mathfrak{F}$ of finite type. If it is proper, then the coarse moduli space is projective.*

Proof. Note that $\mathfrak{F}_{\epsilon,N}$ is an open substack of $\mathfrak{F}$, and hence locally of finite type. The fact that it is bounded (admits a surjective morphism from a scheme of finite type) follows

from [5, § 7]. Assuming properness, the projectivity of the coarse space follows from [6, § 4]. □

Deferring the considerations of finite type, we turn to the valuative criteria for separatedness and properness for $\mathfrak{F}$. To do so, we must understand $\mathbb{Q}$-Gorenstein families of stable log surfaces over DVRs. The following lemma gives a useful characterization of such families.

Let Δ be the spectrum of a DVR with generic point η and special point 0. Let $\pi\colon S \to \Delta$ be a flat, Cohen–Macaulay morphism with reduced geometric fibers of dimension 2 with slc singularities and $D \subset S$ a relative effective Weil divisor.

Lemma 3.9. *In the setup above, assume that S_η has canonical singularities and $\left(S_0, D_{(0)}\right)$ satisfies the index condition. Then $\pi\colon (S, D) \to \Delta$ is a $\mathbb{Q}$-Gorenstein family of log surfaces if and only if both $K_{S/\Delta}$ and D are $\mathbb{Q}$-Cartier.*

Proof. See [19, Proposition 11.7]. The proof goes through verbatim. □

Proposition 3.10 (Valuative criterion of separatedness). *Let $(S_i, D_i) \to \Delta$ for $i = 1, 2$ be $\mathbb{Q}$-Gorenstein families of stable log surfaces satisfying the index condition. Suppose the geometric generic fibers of $S_i \to \Delta$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, 2$. Then an isomorphism between (S_i, D_i) over the generic fiber extends to an isomorphism over Δ.*

Proof. The proof is analogous to the proof of [19, Theorem 2.24]. We recall the salient points.

Possibly after a base change, there exists a common semistable log resolution $(\widetilde{S}, \widetilde{D})$ of (S_i, D_i) for $i = 1, 2$ that is an isomorphism over the generic fiber. Recall that a semistable log resolution is a projective morphism $\widetilde{S} \to S_i$ with the following properties:

   (1) $\widetilde{S}$ is non-singular;
   (2) the exceptional locus of $\widetilde{S} \to S_i$ is a divisor;
   (3) the central fiber $\widetilde{S}_0$ of $\widetilde{S} \to \Delta$ is reduced;

(4) the sum of $\widetilde{S}_0$, the proper transform of D_i, and the exceptional divisors dominating T is a simple normal crossings divisor.

The isomorphism between (S_i, D_i) over the generic fiber implies that the proper transforms of D_i are equal for $i = 1, 2$; call this proper transform $\widetilde{D}$. Let $\epsilon > 0$ be such that the central fibers of $(S_i, D_i) \to \Delta$ satisfy Definition 3.1 with this ϵ (in fact, (S_i, D_i) is klt). Then $i = 1, 2$, the pair (S_i, D_i) is the $K_{\widetilde{S}} + \widetilde{S}_0 + (m/n + \epsilon) \cdot \widetilde{D}$ canonical model of $(\widetilde{S}, \widetilde{D})$. The uniqueness of the log canonical model implies that the isomorphism between (S_i, D_i) over the generic fiber extends over Δ. □

From general principles, we get the following result that partially verifies the valuative criterion of properness for $\mathfrak{F}$.

Proposition 3.11 (A partial valuative criterion of properness). *Let Δ be a DVR with generic point η. Let $(S_\eta, D_\eta) \to \eta$ be a log surface with $S_\eta \cong (\mathbb{P}^1 \times \mathbb{P}^1)_\eta$ and $D_\eta \subset S_\eta$ a smooth curve of bi-degree $(\frac{2n}{m}, \frac{2n}{m})$. Possibly after a base change, there exists a (flat, proper) extension $(S, D) \to \Delta$ of $(S_\eta, D_\eta) \to \eta$ such that the central fiber $(S_0, D_{(0)})$ is a stable log surface and both $K_{S/\Delta}$ and D are $\mathbb{Q}$-Cartier.*

The key missing ingredient in Proposition 3.11 is the assertion that $(S_0, D_{(0)})$ satisfies the index condition, and as a result (thanks to Lemma 3.9) that $(S, D) \to \Delta$ is a $\mathbb{Q}$-Gorenstein family. We do not know an *a priori* reason for the index condition to hold. In the work of Hacking and the present paper, a separate analysis is needed to confirm that it holds in cases of interest.

In subsequent sections, we develop methods to construct (S, D) that yield an explicit description of $(S_0, D_{(0)})$ (see Theorem 6.1 and § 6.6) for stable log quadric surfaces (defined in § 3.5). Thus, for stable log quadrics, Theorem 6.1 subsumes Proposition 3.11 and also verifies the index condition. Nevertheless, we outline the proof of Proposition 3.11 in general, following the proofs of [20, Theorem 2.6] and [20, Theorem 2.12].

Outline of the proof of Proposition 3.11. First, complete (S_η, D_η) to a flat family $(\mathbb{P}^1 \times \mathbb{P}^1, D)$ over Δ. Possibly after a base change on Δ, take a semistable log resolution $(\widetilde{S}, \widetilde{D}) \to (\mathbb{P}^1 \times \mathbb{P}^1, D)$. Run a $K_{\widetilde{S}} + (m/n)\widetilde{D}$ MMP on $(\widetilde{S}, \widetilde{D})$ over Δ (see Example 2.18 for why we can run MMP), resulting in (S_1, D_1). Then run a K_{X_1} MMP on (S_1, D_1) over Δ, resulting in (S_2, D_2). One can show that $(S_2, D_2) \to \Delta$ is a family of semistable log surfaces extending the original family where both K_{S_2} and D_2 are $\mathbb{Q}$-Cartier and $nK_{S_2} + mD_2 \sim 0$. We note one difference at this step from [20, Theorem 2.6]. Since the Picard rank of our generic fiber may not be 1 (unlike the case in [20]), the central fiber of $(S_2, D_2) \to \Delta$ may not be irreducible.

Second, take a maximal crepant blowup $(S_3, D_3) \to (S_2, D_2)$, namely a partial semistable resolution such that the $nK_{S_3} + mD_3$ is the divisorial pullback of $nK_{S_2} + mD_2$, and hence linearly equivalent to 0, and $(S_3, S_3|_0 + (m/n + \epsilon)D_3)$ is dlt for small enough $\epsilon > 0$. Let (S, D) be the $K_{S_3} + (m/n + \epsilon)D_3$ canonical model of (S_3, D_3). Then (S, D) is the required extension. $\qquad\square$

3.5. **Stable log quadrics.** Henceforth, we fix $(m, n) = (2, 3)$. Let $\mathfrak{F}_{K^2=8}$ be the open and closed substack of $\mathfrak{F}$ parametrizing stable log surfaces (S, D) with $K_S^2 = 8$. If S is smooth, then it is a del Pezzo surface with $K_S^2 = 8$ and D is a divisor such that $3K_S + 2D \sim 0$. In particular, K_S is an even divisor class, and hence S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (a smooth quadric in $\mathbb{P}^3$) and D is a divisor of bi-degree $(3, 3)$ on S. Let $\mathfrak{U} \subset \mathfrak{F}_{K^2=8}$ be the open substack that parametrizes stable log surfaces (S, D) with S and D smooth. It is easy to see that $\mathfrak{U}$ is a smooth and irreducible stack of finite type. Indeed, let $U \subset \mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(3, 3))$ be the open subset of the linear series of $(3, 3)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$ parametrizing $D \subset S$ such that D is smooth. Then $\mathfrak{U}$ is the quotient stack $\left[U / \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \right]$.

Definition 3.12 (Stable log quadric)**.** We set $\mathfrak{X}$ as the closure of $\mathfrak{U}$ in $\mathfrak{F}_{K^2=8}$. We call the points of $\mathfrak{X}$ *stable log quadrics*.

Equivalently, a stable log quadric over $\mathbb{K}$ is a pair (S, D) (satisfying the index condition) such that there exists a DVR Δ and a $\mathbb{Q}$-Gorenstein family of stable log surfaces (in the sense of Definition 3.3) whose geometric generic fiber is isomorphic to $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{D})$, where $\mathcal{D} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth curve of bi-degree $(3, 3)$, and whose central fiber is isomorphic to (S, D). By the end of Section 6, we obtain an explicit description of the stable log quadrics. Using this description, we will also see that $\mathfrak{X} \subset \mathfrak{F}_{\epsilon, K^2 = 8}$ for a particular ϵ, and hence it is of finite type.

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4. Trigonal curves and stable log surfaces

The goal of this section is to describe the Tschirnhausen construction, which constructs a semi log canonical surface pair from a degree 3 covering of curves.

Let X and Y be schemes and $\phi \colon X \to Y$ a finite flat morphism of degree 3. Let $E = E_\phi$ be the Tschirnhausen bundle of ϕ. This is the vector bundle on Y defined by the exact sequence

$$(4.1) \qquad\qquad 0 \to \mathcal{O}_Y \to \phi_* \mathcal{O}_X \to E^\vee \to 0.$$

We can associate to ϕ a Cartier divisor $D(\phi) \subset \mathbb{P}E$ whose associated line bundle is $\mathcal{O}_{\mathbb{P}E}(3) \otimes \det E^\vee$. If ϕ is Gorenstein, then $D(\phi)$ is defined as follows. The dual of the quotient map in (4.1) is a map $E \to \phi_* \omega_{X/Y}$, or equivalently a map $\phi^* E \to \omega_{X/Y}$. This map yields an embedding $X \to \mathbb{P}E$ [9]. The divisor $D(\phi)$ is the image of X under this embedding. The construction of $D(\phi)$ extends by continuity to the case where ϕ is not Gorenstein [13, § 4.1]. If $p \in Y$ is a point over which ϕ is not Gorenstein, then $D(\phi)$ contains the entire fiber of $\mathbb{P}E \to Y$ over p. The construction $\phi \rightsquigarrow D(\phi)$ is compatible with arbitrary base-change. Furthermore, it extends to the case where $\phi \colon X \to Y$ is a representable finite flat morphism of degree 3 between algebraic stacks.

Let Y be a reduced stacky curve, and let $\phi \colon X \to Y$ be a representable finite flat morphism of degree 3, étale over the generic points and the singular points of Y. Write

$$D(\phi) = D_H + \pi^* Z,$$

where D_H is finite over Y and $Z \subset Y$ is a divisor. Note that $Z \subset Y$ is supported on the non-Gorenstein locus of ϕ, and in particular on the smooth locus of Y. As we have $X \cong D_H$ over $Y \setminus Z$, we see that D_H is reduced. Let $\phi_H \colon D_H \to Y$ be the natural projection.

Proposition 4.1. *We have the equality* $\operatorname{br} \phi = \operatorname{br} \phi_H + 4Z.$

Proof. It suffices to check the equality of divisors étale locally at a point $y \in Y$. Therefore, we may assume that Y is a scheme. Choose a trivialization $\langle S, T \rangle$ of E around y. We can write $D(\phi)$ as the vanishing locus of a homogeneous cubic

$$f = aS^3 + bS^2T + cST^2 + dT^3,$$

where $a, b, c, d \in \mathcal{O}_{Y,y}$. The discriminant divisor $\mathrm{br}\,\phi$ is cut out by the function

$$\Delta(f) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

Let t be a uniformizer of Y at y and let t^n be the highest power of t that divides a, b, c and d. Then Z is the zero locus of t^n and D_H of the cubic $f_H = f/t^n$. We see that $\Delta(f) = \Delta(f_H) \cdot t^{4n}$, and hence $\mathrm{br}\,\phi = \mathrm{br}\,\phi_H + 4Z$. $\qquad\square$

Let P be an orbi-nodal curve and let $\phi : C \to P$ be an admissible triple cover. Let S be the coarse-space of the surface $\mathbb{P}E_\phi$ and D the coarse space of the divisor $D(\phi) \subset \mathbb{P}E$.

Proposition 4.2. *Suppose* $\mathrm{mult}_p \, \mathrm{br}\,\phi \leq 5$ *for all* $p \in P$. *Then the pair* (S, cD) *is slc for all* $c \leq 7/10$.

Proof. Locally, the pair (S, D) is obtained from the pair $(\mathbb{P}E, D(\phi))$ by taking the quotient by a finite group. Since the property of being slc is preserved under finite group quotients, it suffices to show that $(\mathbb{P}E, cD(\phi))$ is slc.

We first check the slc condition at the singular points of $\mathbb{P}E$. Since $\mathbb{P}E \to P$ is a $\mathbb{P}^1$ bundle, the singular locus of $\mathbb{P}E$ is the pre-image of the singular locus of P. Let $s \in D(\phi) \subset \mathbb{P}E$ lie over a node $p \in P$. Since $C \to P$ is étale over p, étale locally near s the pair $(\mathbb{P}E, D(\phi))$ has the form

(4.2) $$(\mathrm{spec}\,\mathbb{K}[x, y, t]/(xy), t = 0).$$

We see that $(\mathbb{P}E, D(\phi))$ is slc at p.

We now check the slc condition at the smooth points of $\mathbb{P}E$. Let $s \in D(\phi) \subset \mathbb{P}E$ lie over a smooth point $p \in P$. Choose a local coordinate t on P at p and coordinates (y, t) on $\mathbb{P}E$ at s. Recall that we have the decomposition $D(\phi) = D_H + \pi^* Z$. Since $\mathrm{mult}_p \, \mathrm{br} \, \phi \leq 5$, Proposition 4.1 implies that $\mathrm{mult}_p Z \leq 1$.

First, suppose $\mathrm{mult}_p Z = 1$. Then $\mathrm{mult}_p \, \mathrm{br} \, \phi_H \leq 1$; that is, D_H is smooth at s and $D_H \to P$ has at most a simple ramification point at s. In other words, D has the local equation $ty = 0$, which has log canonical threshold 1, or $t(y^2 - t) = 0$, which has log canonical threshold $3/4$; both 1 and $3/4$ are bigger than $7/10$.

Next, suppose $\mathrm{mult}_p Z = 0$. Then $D = D(\phi)$ is flat over p. Let $D^\nu \to D$ be the normalization and $\delta = \mathrm{length}(\mathcal{O}_{D^\nu}/\mathcal{O}_D)$ the delta invariant. It is easy to check ([12, Remark 7.4]) that

$$\mathrm{mult}_p \, \mathrm{br}(D \to P) = \mathrm{mult}_p \, \mathrm{br}(D^\nu \to P) + 2\delta.$$

Since $\mathrm{mult}_p \, \mathrm{br}(D \to P) \leq 5$, we get $\delta \leq 2$. Hence the only possible singularities of D are the A_n singularities for $n \leq 4$. We conclude that the log canonical threshold of D is at most $7/10$, achieved for an A_4 singularity, namely for a D whose equation over P locally over p is $(y^2 - x^5)(y - 1) = 0$. $\qquad\square$

Remark 4.3. We record the observation from the proof of Proposition 4.2 that the only possible singularities of D are A_n for $n \leq 4$.

Remark 4.4. We also record that (S, D) satisfies the index condition. To see this, it suffices to check the singular points of S contained in C. If $p \in S$ is a singular point contained in C, then p is the image of $(0, 0, 0)$ coming from the étale local neighborhood in (4.2). In particular, S has an snc singularity at p, and in particular it is Gorenstein at p.

Let $g \geq 4$. Denote by $\mathcal{H}_g^3$ the Hurwitz space of genus g triple covers of the projective line. More precisely, it is the moduli space whose S-points are given by

$$(P \to S, C \to S, \phi\colon C \to P),$$

where $P \to S$ is a conic bundle, $C \to S$ is a smooth, proper, and connected curve of genus g, and ϕ is a finite flat morphism of degree 3 with simple branching. Let ϵ be a positive number less than $1/30$. Consider the compactification $\overline{\mathcal{H}}_g^3(1/6 + \epsilon)$ of $\mathcal{H}_g^3$ constructed in [12]. We quickly recall its definition and salient properties. The S-points of $\overline{\mathcal{H}}_g^3(1/6 + \epsilon)$ are given by

$$(P \to S, C \to S, \phi\colon C \to P),$$

where $P \to S$ is a (balanced) orbi-nodal curve of genus 0, $C \to S$ is a flat, proper, connected, orbi-nodal curve of genus g, and $\phi\colon C \to P$ is a finite flat morphism satisfying three conditions: (1) ϕ is étale over the nodes of P, (2) the monodromy map $P \setminus \operatorname{br} \phi \to BS_3$ given by ϕ is representable, and (3) the coarse space of P along with the divisor $\operatorname{br} \phi$ with the weight $(1/6 + \epsilon)$ is a weighted stable curve over S in the sense of Hassett [24]. In this definition, a (balanced) orbi-nodal curve over S is a Deligne–Mumford stack whose coarse space is a nodal curve over S, and whose stack structure around a node is given étale locally over the coarse space by

$$[\operatorname{spec} \mathcal{O}_S[x, y]/(xy - s)/\mu_n],$$

for some $s \in \mathcal{O}_S$, where the cyclic group μ_n acts by $\zeta\colon (x, y) \mapsto (\zeta x, \zeta^{-1} y)$ (see [1, 2.1.2]). The main results of [12] imply that $\overline{\mathcal{H}}_g^3(1/6 + \epsilon)$ is a proper and smooth Deligne–Mumford stack over $\mathbb{K}$ [12, Theorem 5.5, Corollary 6.6].

Consider a point $[\phi\colon C \to P]$ of $\overline{\mathcal{H}}_g^3(1/6 + \epsilon)$. Let (S, D) be the pair associated to $C \to P$ by the Tschirnhausen construction. We call (S, D) the *Tschirnhausen pair* of ϕ.

Proposition 4.5. *The divisor $K_S + (2/3 + \epsilon)D$ is ample for all sufficiently small and positive ϵ except in the following cases.*

(1) *$P = \mathbb{P}^1$, and C is a Maroni special curve of genus 4,*

(2) *$P = \mathbb{P}^1$, and $C = \mathbb{P}^1 \cup H$, where H is a hyperelliptic curve of genus 4 attached nodally to $\mathbb{P}^1$ at one point.*

(3) *There is a component $L \cong \mathbb{P}^1$ of P meeting $\overline{P \setminus L}$ in a unique point such that $C \times_P L$ is either*

(a) *a connected curve of arithmetic genus 1, or*

(b) *a disjoint union of L and a connected curve of arithmetic genus 2.*

Recall that a smooth curve C of genus 4 is *Maroni special* if it satisfies the following equivalent conditions: (a) C is not hyperelliptic and lies on a singular quadric in its canonical embedding in $\mathbb{P}^3$, (b) C has a unique g^1_3, (c) there is a degree 3 map $\phi \colon C \to \mathbb{P}^1$ such that the Tschirnhausen bundle $(\phi_* \mathcal{O}_C / \mathcal{O}_{\mathbb{P}^1})^\vee$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(4)$. In contrast, a *Maroni general* C of genus 4 (a) is non-hyperelliptic and lies on on a smooth quadric in its canonical embedding in $\mathbb{P}^3$, (b) has two distinct g^1_3's, and (c) has Tschirnhausen bundle isomorphic to $\mathcal{O}(3) \oplus \mathcal{O}(3)$.

Proof of Proposition 4.5. The numerical criteria of ampleness may be checked on the stack, rather than the coarse space. Therefore, in the rest of the proof, let S denote the stack $\mathbb{P}E_\phi$ and $D(\phi) \subset S$ the Tschirnhausen divisor associated to ϕ. As the coarse space map of $\mathbb{P}E_\phi$ is unramified in codimension one, the divisor classes remain unchanged.

It suffices to check ampleness on each irreducible component of S. Let L be an irreducible component of P. Set $C_L = L \times_P C$, let $\phi_L \colon C_L \to L$ be the restriction of ϕ, and let E_L be the Tschirnhausen bundle of ϕ_L. Set $S_L = \mathbb{P}E_L$ and $D_L = D \cap S_L$. Let $n = \deg E_L$, so that $2n = \deg \operatorname{br} \phi_L$.

We know that the Neron-Severi group of S_L is spanned by the class F of a fiber and the class ζ of $\mathcal{O}_{\mathbb{P}E}(1)$. The intersection form is determined by $F^2 = 0$, $\zeta F = 1$, and

$\zeta^2 = n$. The cone of curves on S_L is spanned by F and the class of a section σ. Let m be the number of points in $L \cap \overline{(P \setminus L)}$ counted without any multiplicity. Then, it is easy to check that

$$\deg K_P|_L = -2 + m.$$

Therefore, we obtain that

$$K_S|_{S_L} \sim (m + n - 2)F - 2\zeta.$$

We also have

$$D_L \sim 3\zeta - nF.$$

Therefore, we get

$$K_S + (2/3 + \epsilon)D\big|_{S_L} \sim (m + n/3 - 2)F + \epsilon(3\zeta - nF).$$

We see immediately that $(K_S + (2/3 + \epsilon)D) \cdot F = 3\epsilon > 0$. Thus, it remains to check that $(K_S + (2/3 + \epsilon)D) \cdot \sigma > 0$ for the extremal section σ.

If $m + n/3 > 2$, then it is clear that $(K_S + (2/3 + \epsilon)D) \cdot \sigma > 0$ for small enough ϵ. As a result, we only need to consider the cases where $m \leq 2$. In fact, the case $m = 2$ is also easy to dispose off. If $m = 2$, then the ampleness of $K_P + (1/6 + \epsilon)\operatorname{br}\phi$ implies that $n > 0$, and hence $m + n/3 > 2$.

We now consider the cases $m = 0$ and $m = 1$. First, suppose $m = 0$. Then $n = g + 2 \geq 6$, so $m + n/3 \geq 2$, with equality only if $g = 4$. If $g = 4$, then E is isomorphic to either $\mathcal{O}(3) \oplus \mathcal{O}(3)$, or $\mathcal{O}(2) \oplus \mathcal{O}(4)$, or $\mathcal{O}(1) \oplus \mathcal{O}(5)$. For $E \cong \mathcal{O}(3) \oplus \mathcal{O}(3)$, it is easy to check that $K_S + (2/3 + \epsilon)D$ is ample. The cases $E \cong \mathcal{O}(2) \oplus \mathcal{O}(4)$ and $E \cong \mathcal{O}(1) \oplus \mathcal{O}(5)$ yield the possibilities (1) and (2), respectively, in the statement of Proposition 4.5.

Next, suppose $m = 1$. The ampleness of $K_P + (1/6 + \epsilon)\operatorname{br}\phi$ implies that $n = 3$. Let $p \in L$ be the unique point of intersection of L with $\overline{P \setminus L}$. We know that vector bundles on L split as direct sums of line bundles, and line bundles on L are classified by their

degree [33]. Note that the degree of a line bundle is not necessarily an integer, but an element of $\frac{1}{d}\mathbb{Z}$, where d is the order of $\mathrm{Aut}_p L$. Suppose

$$E_L \cong \mathcal{O}_L(a) \oplus \mathcal{O}_L(b),$$

where $a, b \in \frac{1}{d}\mathbb{Z}$ with $0 \leq a \leq b$ and $a + b = n$. The extremal section σ is given by $\sigma \sim \zeta - bF$. Since $C \to P$ is an admissible triple cover, d is either 1, 2, or 3. If $d = 1$, then (a, b) is either $(1, 2)$ or $(0, 3)$. These two cases yield the possibilities (3a) and (3b), respectively, in the statement of Proposition 4.5.

It remains to consider the cases $d = 2$ and $d = 3$. Consider the map $\underline{\phi}_L \colon \underline{C}_L \to \underline{L}$ on coarse spaces associated to $\phi_L \colon C_L \to L$. Since $d > 1$, we know that C_L is not isomorphic to its coarse space $\underline{C}_L$, and hence E_L is not pulled back from $\underline{L}$. Said differently, a and b are not both integers. We have $\deg \mathrm{br}\, \underline{\phi}_L = \deg \mathrm{br}\, \phi_L + (d - 1) = 2n + d - 1$, and $\deg \mathrm{br}\, \underline{\phi}_L$ must be even. So we cannot have $d = 2$. For $d = 3$, observe that $\underline{C}_L$ must be totally ramified over p. We compute that

$$K_S + (2/3 + \epsilon)D\big|_{S_L} \cdot \sigma = \epsilon(3\zeta - nF) \cdot \sigma$$

$$= \epsilon(2a - b).$$

Since $\underline{C}_L$ is triply ramified over p, it is locally irreducible over p. As a result, D_L does not contain σ as a component. We conclude that $D_L \cdot \sigma = 2a - b \geq 0$. The further constraints that $a + b = n$ and that not both a and b are integers force $2a - b > 0$. As a result, we get that $K_S + (2/3 + \epsilon)D$ is in fact ample. $\qquad\square$

4.1. **Stable and unstable pairs in genus 4.** Let $g = 4$, and $[\phi \colon C \to P] \in \overline{\mathcal{H}}_g^3(1/6 + \epsilon)$. Let (S, D) be the pair associated to $C \to P$ by the Tschirnhausen construction.

Proposition 4.6. *The pair (S, D) is a semi-stable log quadric surface. It is also stable except in the cases enumerated in Proposition 4.5*

Proof. By Proposition 4.2, $(S, 2/3 \cdot D)$ is slc. By Proposition 4.5, there exists $\epsilon > 0$ such that $K_S + (2/3 + \epsilon)D$ is ample, except in the listed cases. It remains to show that $3K_S + 2D$ is linearly equivalent to 0. It suffices to show this on the stack $\mathbb{P}E_\phi$. We have

$$(4.3) \qquad\qquad K_{\mathbb{P}E_\phi} \cong \mathcal{O}(-2) \otimes \pi^* \det E_\phi \otimes \pi^* K_P$$

where $\pi \colon \mathbb{P}E_\phi \to P$ is the natural projection. By construction, we have

$$(4.4) \qquad\qquad \mathcal{O}(D) \cong \mathcal{O}(3) \otimes \pi^* \det E_\phi^\vee.$$

Observe that $2 \det E_\phi$ is the branch divisor B of $C \to P$. Furthermore, see that we always have

$$(4.5) \qquad\qquad K_P + 1/6 \cdot B \sim 0.$$

To check this, note that we either have $P \cong \mathbb{P}^1$ or $P \cong P_1 \cup P_2$ with the 12 points of B separated as 6+6 on the two components. In both cases, (4.5) holds. From (4.3), (4.4), and (4.5), we get that $3K_{\mathbb{P}E_\phi} + 2D \sim 0$. $\qquad\square$

We enumerate the strictly semi-stable and stable cases for genus 4. Recall that ϵ is such that $0 < \epsilon < 1/30$.

Stable pairs: A $(1/6 + \epsilon)$-admissible cover $\phi \colon C \to P$ yields a stable log quadric surface (S, D) in the following cases.

(1) $P \cong \mathbb{P}^1$ and $\phi \colon C \to P$ is Maroni general in the sense that $E_\phi \cong \mathcal{O}(3) \oplus \mathcal{O}(3)$. In this case, we see that $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $D \subset S$ is a divisor of bi-degree $(3, 3)$.

(2) $P = P_1 \cup_s P_2$ is a twisted curve with two smooth irreducible components P_1 and P_2 attached nodally at s. Both components are rational (their coarse spaces are $\mathbb{P}^1$), and the only point with a non-trivial automorphism group

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on P is the node s with $\mathrm{Aut}_s P = \mu_3$. The curve C is schematic, and is isomorphic to $C_1 \cup_p C_2$, where $p \in C$ is a node and C_i has arithmetic genus 2 for each i. The map ϕ restricts to a degree 3 map $C_i \to P_i$, étale over s, and p is the unique pre-image of s. In this case, we see that S is the coarse space of a projective bundle $\mathcal{P}(\mathcal{O}(5/3, 4/3) \oplus \mathcal{O}(4/3, 5/3))$, where $\mathcal{O}(a_1, a_2)$ is a line bundle on P whose restriction to P_i is $\mathcal{O}(a_i)$. Let S_1 and S_2 be the two components of S over coarse spaces of P_1 and P_2, respectively. Then $D \cap S_i \subset S_i$ is a divisor of class $3\sigma_i + 2F$ where $\sigma_i \subset S_i$ is the image of the unique section of $\mathcal{P}(\mathcal{O}(5/3) \oplus \mathcal{O}(4/3))$ of self-intersection $(-1/3)$. Furthermore, D intersects the double curve $S_1 \cap S_2$ transversely.

Unstable pairs: A $(1/6 + \epsilon)$-admissible cover $\phi : C \to P$ yields a semi-stable but not stable log quadric surface (S, D) in the following cases.

(1) $P \cong \mathbb{P}^1$, and $\phi : C \to P$ is Maroni special. In this case, $S \cong \mathbb{F}_2$ and $D \subset S$ is a divisor of class $3\sigma + 6F$, where $\sigma \subset S$ is the directrix.

(2) $P \cong \mathbb{P}^1$ and $C \cong H \cup_p L$, where $L \cong \mathbb{P}^1$, and H is a curve of arithmetic genus 4 attached nodally to L at one point p. The map ϕ restricts to a degree 2 map $H \to P$ and to a degree 1 map $L \to P$. In this case, $S \cong \mathbb{F}_4$ and D is the union of σ and a divisor of class $2\sigma + 9F$.

(3) $P \cong P_1 \cup_s P_2$, with $P_i \cong \mathbb{P}^1$ attached nodally at a point s; and $C \cong C_1 \cup_{p,q,r} C_2$, where $p, q, r \in C$ are nodes and C_i has arithmetic genus one for each i. The map ϕ restricts to a degree 3 map $C_i \to P_i$, étale over s, and $\{p, q, r\}$ is the pre-image of s. These cases break into three further subcases. In all three subcases, we have $S = S_1 \cup S_2$ and $D = D_1 \cup D_2$. The subcases are:

  (a) For $i = 1, 2$, we have $C_i = H_i \sqcup L_i$, where $L_i \cong \mathbb{P}^1$ and H_i is a connected curve of genus 2. The map $\phi : C_i \to P_i$ restricts to a degree 2 map on H_i and to a degree 1 map on L_i. L_1 and L_2 do not intersect as C is connected. In this case, we have $S_i \cong \mathbb{F}_3$; and $D_i = \sigma_i \sqcup H_i \subset S_i$,

where $\sigma_i \subset S_i$ is the unique section of self-intersection (-3) and $H_i \subset S_i$ is a divisor of class $2\sigma_i + 6F$ intersecting the fiber $S_1 \cap S_2$ transversely.

(b) For $i = 1, 2$, the curve C_i is a connected curve of arithmetic genus 1. In this case, we have $S_i \cong \mathbb{F}_1$, and $D_i \subset S_i$ is a divisor of class $3\sigma_i + 3F$ intersecting the fiber $S_1 \cap S_2$ transversely, where σ_i is the unique section of self-intersection (-1) in S_i.

(c) C_1, S_1, D_1 are as in case (3a) and C_2, S_2, D_2 are as in case (3b).

5. FLIPS

The goal of this section is to describe two kinds of flips that are necessary for the stable reduction of log surfaces arising from trigonal curves. The first involves flipping a -4 curve and the second a -3 curve on the central fiber in a family of surfaces.

5.1. **Flipping a** (-4) **curve (Type I flip).** Let Δ be the spectrum of a DVR. Let $\mathcal{X} \to \Delta$ be a smooth, but not necessarily proper, family of surfaces. Let $\mathcal{D} \subset \mathcal{X}$ an effective divisor flat over Δ with a non-singular general fiber. Denote by (X, D) the special fiber of $(\mathcal{X}, \mathcal{D}) \to \Delta$. Suppose (X, D) has the following form. We have $D = \sigma \cup C$, where $\sigma \subset X$ is a -4 curve and $C \subset X$ is a non-singular curve that intersects σ transversely at one point p. In this case, we can see that $(\mathcal{X}, \mathcal{D})$ has canonical singularities.

The leftmost quadrilateral in Figure 1 is our diagrammatic representation of X along with the configuration of curves the C and σ on it. In general, we represent surfaces by plane polygons, and depict curves lying on the surface along the edges or on the interior. An number next to an edge, if any, is the self-intersection of the curve represented by the edge. Descriptive text next to a point is the description of the singularity at that point.

Construct (X', D') from (X, D) as follows (see Figure 1). Let $\widetilde{X} \to X$ be the blow up of X two times, first at p (the intersection point of C and σ), and second at the intersection point of the exceptional divisor E_1 of the first blow-up with the proper transform of C. Equivalently, $\widetilde{X}$ is the minimal resolution of the blow-up of X at the unique subscheme of C of length 2 supported at p. Denote by $\widetilde{C} \subset \widetilde{X}$ and $\widetilde{\sigma} \subset \widetilde{X}$ the proper transforms of σ and C, and by $E_i \subset X$ the proper transform of the exceptional divisor of the ith blow up, for $i = 1, 2$. On $\widetilde{X}$, the curves $(E_1, \widetilde{\sigma})$ form a chain of rational curves of self-intersections $(-2, -5)$. Let $\widetilde{X} \to X'$ be the contraction of this chain. Then the surface X' is smooth everywhere except at the image point of the rational chain, where it has the quotient singularity $\frac{1}{9}(1, 2)$. Let $C' \subset X'$ be the image of $\widetilde{C}$. Set $D' = C'$.

Figure 1 is our diagrammatic representation of the transformation from X to X'.

Proposition 5.1. *Let $(\mathcal{X}, \mathcal{D}) \to \Delta$ be a family of log surfaces as described above. Then there exists a flat family $(\mathcal{X}', \mathcal{D}') \to \Delta$ isomorphic to $(\mathcal{X}, \mathcal{D})$ over Δ° such that the central fiber of $(\mathcal{X}', \mathcal{D}') \to \Delta$ is (X', D'). Furthermore, the threefold $\mathcal{X}'$ is $\mathbb{Q}$-factorial and has canonical singularities.*

Remark 5.2. Note that (X', D') is log canonical. Also, it is important to observe that it depends only on (X, D), not on the family $(\mathcal{X}, \mathcal{D}) \to \Delta$.

The rest of § 5.1 is devoted to the proof of Proposition 5.1. In the proof, we construct $\mathcal{X}'$ from $\mathcal{X}$ by an explicit sequence of birational transformations. We divide these birational transformations into two stages. The first stage consists of a sequence of blow-ups along -4 curves. The second stage consists of a sequence of a particular kind of flip, which we call a *topple*. We begin by studying blow ups and topples.

5.1.1. *A (-4)-blow up.* Let $(\mathcal{X}, \mathcal{D}) \to \Delta$ be as in the statement of Proposition 5.1. Let $\beta : \widetilde{\mathcal{X}} \to \mathcal{X}$ be the blow up along σ. Let $\widetilde{\mathcal{D}}$ be the proper transform of $\mathcal{D}$ in $\widetilde{\mathcal{X}}$ and $E \subset \widetilde{\mathcal{X}}$ the exceptional divisor. The central fiber of $\widetilde{\mathcal{X}} \to \Delta$ is the union of E and the proper transform of X, which is an isomorphic copy of X. We know that E is the projectivization of the normal bundle of σ in $\mathcal{X}$. The next lemma identifies the normal bundle.

Lemma 5.3. *The normal bundle $N_{\sigma/\mathcal{X}}$ is given by*

$$
N_{\sigma/\mathcal{X}} \cong \begin{cases} \mathcal{O}(-1) \oplus \mathcal{O}(-3) & \text{if } \mathcal{D} \text{ is non-singular,} \\ \mathcal{O} \oplus \mathcal{O}(-4) & \text{otherwise.} \end{cases}
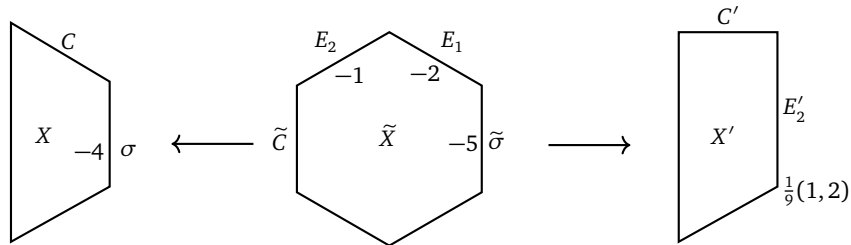$$

FIGURE 1. The central fiber X is replaced by X' in a type 1 flip.

In the first case, we have $E \cong \mathbb{F}_2$, and $E \cap \widetilde{\mathcal{D}}$ is the unique -2 curve on E. In the second case, we have $E \cong \mathbb{F}_4$, and $E \cap \widetilde{\mathcal{D}}$ is the union of the unique -4 curve on E and a fiber F of $E \to \mathbb{P}^1$.

Proof. We have the exact sequence of bundles

$$0 \to N_{\sigma/X} \to N_{\sigma/\mathfrak{X}} \to N_{X/\mathfrak{X}}\big|_\sigma \to 0.$$

In this sequence, the kernel is $\mathcal{O}(-4)$ and the cokernel is $\mathcal{O}$. Therefore, the only possibilities for $N_{\sigma/\mathfrak{X}}$ are $\mathcal{O}(-i) \oplus \mathcal{O}(-4+i)$ for $i = 0, 1, 2$. We must now rule out $i = 2$, and characterize the remaining two.

The divisor class $[\widetilde{\mathcal{D}}]$ is given by

$$[\widetilde{\mathcal{D}}] = [\beta^*\mathcal{D}] - [E].$$

Since $\widetilde{\mathcal{D}}$ intersects E properly, the restriction $\widetilde{\mathcal{D}}|_E$ must be effective. An easy calculation shows that $\left(\widetilde{\mathcal{D}}|_E\right)^2 = -2$. Now, $\mathbb{P}(\mathcal{O}(-2) \oplus \mathcal{O}(-2)) = \mathbb{P}^1 \times \mathbb{P}^1$ contains no effective classes of self-intersection -2. Therefore, we can rule out the possibility of $i = 2$, namely the possibility that $N_{\sigma/\mathfrak{X}} \cong \mathcal{O}(-2) \oplus \mathcal{O}(-2)$.

For the remainder, we examine the map $\widetilde{\mathcal{D}} \to \mathcal{D}$, which is the blow-up along σ, and the curve $E \cap \widetilde{\mathcal{D}}$. Since the central fiber of $\mathcal{D} \to \Delta$ is a nodal curve with the node at p, the only possible singularity of $\mathcal{D}$ is at p. Hence, the curve $E \cap \widetilde{\mathcal{D}}$ contains a unique reduced component $\widetilde{\sigma}$ mapping isomorphically to σ, and possibly some other components that are contracted to p. As a divisor on E, we may write

$$E \cap \widetilde{\mathcal{D}} = s + m \cdot f,$$

for some $m \geq 0$, where s is a section of $E \to \sigma$ and f is the fiber of $E \to \sigma$ over p.

Suppose $\mathcal{D}$ is non-singular. Then the blow-up $\widetilde{\mathcal{D}} \to \mathcal{D}$ is an isomorphism, and therefore we have $m = 0$. As a result, we see that $E \to \sigma$ has a section of self-intersection (-2). We

conclude that $N_{\sigma/\mathcal{X}} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-3)$, and $E \cap \widetilde{D}$ is the unique section of self-intersection (-2).

Suppose $\mathcal{D}$ is singular. Then it has an A_n-singularity at p for some $n \geq 1$. In that case, $\widetilde{D} \to D$ contracts a $\mathbb{P}^1$. Therefore, we must have $m > 0$. Since $\mathbb{F}_2$ does not contain a class of the form $s + m \cdot f$ of self-intersection (-2), we can rule out $N_{\sigma/\mathcal{X}} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-3)$, and get $N_{\sigma/\mathcal{X}} \cong \mathcal{O} \oplus \mathcal{O}(-4)$. The unique effective class of the form $s + m \cdot f$ on $E \cong \mathbb{F}_4$ is the union of the section of self-intersection (-4) and a fiber. $\qquad \square$

5.1.2. *A topple.* Let $\mathcal{Z} \to \Delta$ be a flat and generically smooth family of surfaces with central fiber $\mathcal{Z}_0 = S \cup T$. Let $\mathcal{D} \subset \mathcal{Z}$ be a non-singular surface such that the central fiber $\mathcal{D}_0$ of $\mathcal{D} \to \Delta$ has the form $\mathcal{D}_0 = C \cup \sigma$, where C lies on S and σ lies on T. We require that the configuration of S, T, C, and σ is as shown in the leftmost diagram in Figure 2. More precisely, we assume the following.
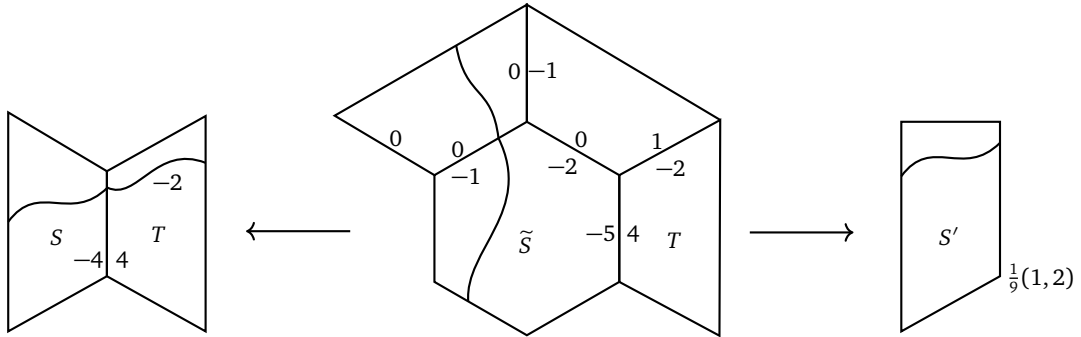
FIGURE 2. The central fibers in a topple.

(1) The surfaces S and T meet transversely along a curve $B \cong \mathbb{P}^1$. In particular, S and T are non-singular along B.

(2) Both C and σ are non-singular, and T is non-singular along σ.

(3) B has self-intersection (-4) on S and 4 on T.

(4) σ has self-intersection (-2) on T.

(5) On S, the curves C and B intersect transversely at a unique point p. Similarly, on T, the curves σ and B intersect transversely at the same point p.

(6) The Neron-Severi group $NS(T)$ is spanned by B and σ.

We make two additional assumptions on the threefold $\mathcal{Z}$. First, assume that we have a projective morphism $\pi\colon \mathcal{Z} \to \mathcal{Y}$ that is an isomorphism on the general fiber and contracts T to a point. Second, assume that $\mathcal{Z}$ is non-singular along B, C, and σ, and has canonical singularities elsewhere.

Lemma 5.4. *In the setup above, there exists a family of log surfaces $(\mathcal{Z}', \Delta') \to \Delta$ isomorphic to $(\mathcal{Z}, \Delta) \to \Delta$ on the generic fiber, whose central fiber (S', C') is obtained from (S, C) by the procedure $(X, C) \rightsquigarrow (X', C')$ described in Figure 1 with the role of σ played by B. Furthermore, the threefold $\mathcal{Z}'$ is $\mathbb{Q}$-factorial, and the surface $\mathcal{D}'$ is non-singular.*

We say that the transformation $\mathcal{Z} \dashrightarrow \mathcal{Z}'$ is a *topple* along T.

Proof. We construct $\mathcal{Z}'$ from $\mathcal{Z}$ by two blow ups and two blow downs.

Let $\mathcal{Z}^1 \to \mathcal{Z}$ be the blow up of $\mathcal{Z}$ along σ; let $E^{(1)} \subset \mathcal{Z}^{(1)}$ be the exceptional divisor; and let $\sigma^{(1)} \subset E^{(1)}$ be the intersection of $E^{(1)}$ with the proper transform $\mathcal{D}^{(1)}$ of $\mathcal{D}$. From an easy computation, we get that $N_{\sigma/\mathcal{Z}} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-2)$, and hence $E^{(1)} \cong \mathbb{F}_1$, and $\sigma^{(1)} \subset E^{(1)}$ is the directrix.

Let $\mathcal{Z}^{(2)} \to \mathcal{Z}^{(1)}$ be the blow up of $\mathcal{Z}^{(1)}$ along $\sigma^{(1)}$. Define $E^{(2)}$, $\mathcal{D}^{(2)}$, and $\sigma^{(2)}$ as before. By similar computation as above, it follows that $E^{(2)} \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\sigma^{(2)} \subset E^{(2)}$ is a ruling line, more precisely, a line of the ruling opposite to the fibers of $E^{(2)} \to \sigma^{(1)}$. The middle picture in Figure 2 shows a sketch of the central fiber $\mathcal{Z}_0^{(2)}$ of $\mathcal{Z}^{(2)} \to \Delta$.

Let $\mathcal{Z}^{(2)} \to \mathcal{Z}^{(3)}$ be the contraction in which the lines of the ruling $\left[\sigma^{(2)}\right]$ are contracted. Note that this contractions contracts $E^{(2)}$ to a $\mathbb{P}^1$, but in the opposite way compared to the contraction $\mathcal{Z}^{(2)} \to \mathcal{Z}^{(1)}$. We can show that the contraction $\mathcal{Z}^{(2)} \to \mathcal{Z}^{(3)}$ exists by Theorem 2.12, as the curve $\sigma^{(2)}$ spans a $K_{\mathcal{Z}^{(2)}}$ negative ray in $\mathrm{NE}(\pi)$. This contraction must contract all the ruling lines in the same class as $\sigma^{(2)}$, and therefore must contract E to a $\mathbb{P}^1$. This is a divisorial contraction, and hence $\mathcal{Z}^{(3)}$ is $\mathbb{Q}$-factorial with canonical singularities. It is easy to identify this divisorial contraction: since E is isomorphic to

$\mathbb{P}^1 \times \mathbb{P}^1$ and the contracted extremal curve is a ruling, this divisorial contraction is as in Theorem 2.13(1), and hence $\mathcal{Z}^{(3)}$ is in fact non-singular along the image of E. Let $\mathcal{D}^{(3)} \subset \mathcal{Z}^{(3)}$ be the image of $\mathcal{D}^{(2)}$. The images of $E^{(1)}$ and T in $\mathcal{Z}^{(3)}$ lie away from $\mathcal{D}^{(3)}$. The image $\overline{E}^{(1)}$ of $E^{(1)}$ is isomorphic to $\mathbb{P}^2$. The image of T is isomorphic to T; we denote it by the same letter. Let $\mathcal{Z}^{(3)} \to \mathcal{Z}^{(4)}$ be the contraction that maps $\overline{E}^{(1)}$ to a point. This is the contraction of the $K_{\mathcal{Z}^{(4)}}$-negative extremal ray of $\mathrm{NE}(\pi)$ spanned by a line in $\overline{E}^{(1)}$. The image $\overline{T}$ of T in $\mathcal{Z}^{(4)}$ is a surface of Picard rank 1; the only curve class on it is $[B]$. Since the contraction is divisorial, $\mathcal{Z}^{(4)}$ is $\mathbb{Q}$-factorial with canonical singularities. As before, it is easy to identify this divisorial contraction: since $\overline{E}^{(1)}$ is isomorphic to $\mathbb{P}^2$, the threefold $\mathcal{Z}^{(3)}$ is non-singular along $\overline{E}^{(1)}$, and the normal bundle of $\overline{E}^{(1)}$ is isomorphic to $\mathcal{O}(-2)$, the divisorial contraction is as in Theorem 2.13(5), and $\mathcal{Z}^{(4)}$ has the quotient singularity $\mathbb{A}^3/\mathbb{Z}_2$ at the image point of $\overline{E}^{(1)}$. Finally, let $\mathcal{Z}^{(4)} \to \mathcal{Z}'$ be the contraction that maps $\overline{T}$ to a point. It is the contraction of the $K_{\mathcal{Z}^{(4)}}$-negative extremal curve $[B]$ of $\mathrm{NE}(\pi)$. The rightmost picture in Figure 2 shows a sketch of the central fiber $\mathcal{Z}'_0$ of $\mathcal{Z}' \to \Delta$.

Set $S' = \mathcal{Z}'_0$ and $C' = \mathcal{D}'_0$. The threefold $\mathcal{Z}'_0$ is canonical, and hence Cohen–Macaulay. Theerfore, S' is also Cohen-Macaulay, and since it is non-singular in codimension 1, it is normal. Observe that the transformation from S to S' is exactly as described in Figure 1—two blow ups on C followed by the contraction of a $(-2, -5)$ chain of $\mathbb{P}^1$s, resulting in a $\frac{1}{9}(1, 2)$ singularity. $\qquad \square$

Remark 5.5. In the notation of Lemma 5.4, note that the Picard rank of S' is the same as the Picard rank of S, and the self intersection of C' on S' is given in terms of the self-intersection of C on S by

$$C'^2 = C^2 - 2.$$

5.1.3. *Proof of Proposition 5.1.* Having described the two required birational transformations, we take up the proof of Proposition 5.1.

The transformation from $(\mathfrak{X}, \mathcal{D})$ to $(\mathfrak{X}', \mathcal{D}')$ goes through a number of intermediate steps $(\mathfrak{X}^{(i)}, \mathcal{D}^{(i)})$, which can be divided into two stages. Throughout, $\mathcal{D}^{(i)} \subset \mathfrak{X}^{(i)}$ denotes the closure of $\mathcal{D}\big|_{\Delta^\circ}$ in $\mathfrak{X}^{(i)}$.

Let $\pi \colon \mathfrak{X} \to \mathcal{Y}$ be the contraction of the curve σ. All the intermediate steps $\mathfrak{X}^{(i)}$ will be projective over $\mathcal{Y}$. We use the letter π to denote the obvious map from various spaces to $\mathcal{Y}$.

Stage 1 (Blowups): Set $\mathfrak{X}^{(0)} = \mathfrak{X}$, $E_0 = X$, $\mathcal{D}^{(0)} = \mathcal{D}$, and $\sigma^{(0)} = \sigma$. For a Hirzebruch surface $E \cong \mathbb{F}_k$ for $k \geq 1$, denote by σ^E the directrix, namely the unique section of self-intersection $(-k)$.

Suppose $\mathcal{D}^{(0)}$ has an A_n singularity at p, where $n \geq 1$. Let $\mathfrak{X}^{(1)} = \mathrm{Bl}_\sigma \mathfrak{X}$. Denote by $E_1 \subset \mathfrak{X}^{(1)}$ the exceptional divisor and $\mathcal{D}^{(1)}$ the proper transform of $\mathcal{D}^{(0)}$. By Lemma 5.3, we have $E_1 \cong \mathbb{F}_4$ and $\mathcal{D}^{(1)} \cap E_1 = \sigma^{E_1} \cup F$. Set $\sigma^{(1)} = \sigma^{E_1}$. Note that Lemma 5.3 applies to $\sigma^{(1)} \subset \mathfrak{X}^{(1)}$ and its blow up. Indeed, the conditions on the central fiber of $(\mathfrak{X}, \mathcal{D})$ hold for $(\mathfrak{X}^{(1)}, \mathcal{D}^{(1)})$ in an open subset around the -4 curve $\sigma^{(1)}$ (the role of σ is played by $\sigma^{(1)}$, and the role of C by F). Note that after the blowup, $\mathcal{D}^{(1)}$ has an A_{n-1} singularity. Continue blowing up the -4 curves in this way, obtaining a sequence

$$\mathfrak{X}^{(n)} \to \cdots \to \mathfrak{X}^{(1)} \to \mathfrak{X}^{(0)}.$$

Figure 3a shows the central fiber of $\mathfrak{X}^{(n)} \to \Delta$. In this figure, some curves are labelled with two numbers. Note that these curves lie on two surfaces; the two numbers are the self-intersection numbers of the curve on either surface.

We now continue from $\mathfrak{X}^{(n)}$ and the non-singular surface $\mathcal{D}^{(n)}$, where $n \geq 0$. Let $\mathfrak{X}^{(n+1)} \to \mathfrak{X}^{(n)}$ be the blow up of the -4 curve $\sigma^{(n)} \subset \mathfrak{X}^{(n)}$. By Lemma 5.3, the exceptional divisor E_{n+1} is isomorphic to $\mathbb{F}_2$ and it intersects the proper transform $\mathcal{D}^{(n+1)}$ of $\mathcal{D}^{(n)}$ in the unique (-2) curve $\sigma^{E_{n+1}}$. Set $\sigma^{(n+1)} = \sigma^{E_{n+1}}$. Figure 3b shows the central fiber of $\mathfrak{X}^{(n+1)} \to \Delta$.

(A) The central fiber of $\mathfrak{X}^{(n)} \to \Delta$

(B) The central fiber of $\mathfrak{X}^{(n+1)} \to \Delta$
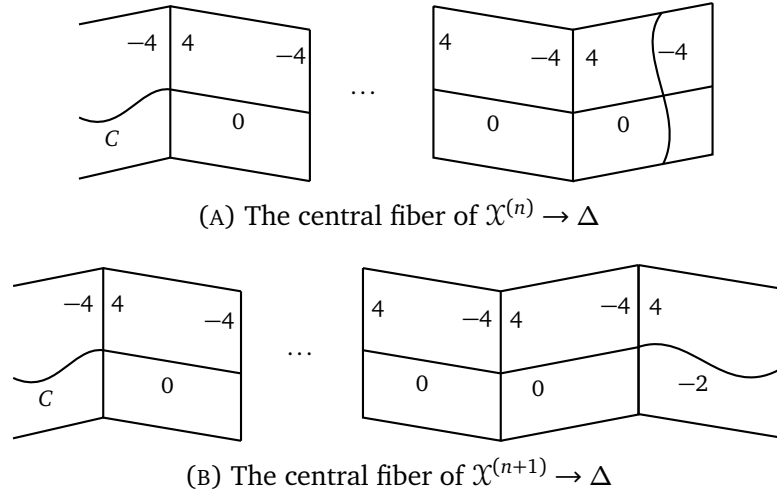
FIGURE 3. The central fibers of the nth and the $(n+1)$th blow up

Stage 2 (Topples): We now continue with $\mathfrak{X}^{(n+1)}$, whose central fiber is the union

$$\mathfrak{X}_0^{(n+1)} = E^{(0)} \cup \cdots \cup E^{(n)} \cup E^{(n+1)}.$$

After restricting to an open set containing $E^{(n+1)}$, we see that we can topple $\mathfrak{X}^{(n+1)}$ along $E^{(n+1)}$. That is, the family $(\mathfrak{X}^{(n+1)}, \mathcal{D}^{(n+1)}) \to \Delta$ satisfies the assumptions of Lemma 5.4. Let $\mathfrak{X}^{(n+1)} \dashrightarrow \mathfrak{X}^{(n+2)}$ be the topple along $E^{(n+1)}$. Denote by $E^{(n+2)}$ (resp. $\mathcal{D}^{(n+2)}$) the image of $E^{(n)}$ (resp. $\mathcal{D}^{(n+1)}$) under the topple. Then the central fiber of $\mathfrak{X}^{(n+2)}$ is the union

$$\mathfrak{X}_0^{(n+2)} = E^{(0)} \cup \cdots \cup E^{(n-1)} \cup E^{(n+2)}.$$

See Figure 4 for a sketch of this configuration.

FIGURE 4. The central fibers in one step of the sequence of topples

We observe again that an open subset containing $E^{(n+2)}$ satisfies the assumptions of Lemma 5.4, and we continue the process by toppling $\mathfrak{X}^{(n+2)}$ along $E^{(n+2)}$. After $(n+1)$

topples, we arrive at a pair $(\mathcal{X}', \mathcal{D}') = (\mathcal{X}^{(2n+2)}, \mathcal{D}^{(2n+2)})$. Note that in the very first topple, the toppled surface $E^{(n+1)}$ is isomorphic to $\mathbb{F}_2$. In the subsequent topple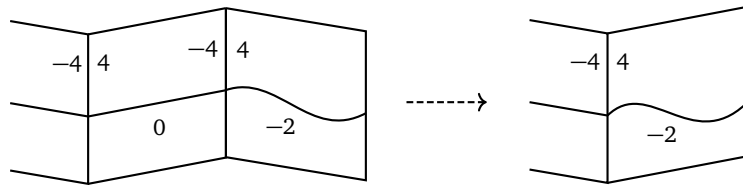s, however, the toppled surface is different—it is a rational surface of Picard rank 2 with a $\frac{1}{9}(1,2)$ singularity (the singularity is not shown in Figure 4).

By construction, $\mathcal{X}'$ is $\mathbb{Q}$-factorial with canonical singularities. In particular, both $\mathcal{D}'$ and $K_{\mathcal{X}'}$ are $\mathbb{Q}$-Cartier. By construction, the central fiber of $(\mathcal{X}', \mathcal{D}') \to \Delta$ is (X', D'). The proof of Proposition 5.1 is now complete.

5.2. **Flipping a** (-3) **curve (Type II flip).** Let Δ be the spectrum of a DVR. Let $\mathcal{X} \to \Delta$ be a flat family of surfaces and $\mathcal{D} \subset \mathcal{X}$ a divisor flat over Δ. Assume that both $\mathcal{X} \to \Delta$ and $\mathcal{D} \to \Delta$ are smooth over Δ°. Suppose the central fiber (X, D) of $(\mathcal{X}, \mathcal{D}) \to \Delta$ has the following form: X is reduced and has two non-singular irreducible components S, T, which meet transversely along a non-singular curve B, and $D = C \cup \sigma$, where $\sigma \subset T$ is a (-3)-curve that meets B transversely at a point p and $C \subset S$ is a non-singular curve that meets B transversely at the same point p. Recall that a (-3)-curve is a curve isomorphic to $\mathbb{P}^1$ whose self-intersection is -3. The left-most diagram in Figure 5 shows a sketch of (X, D).
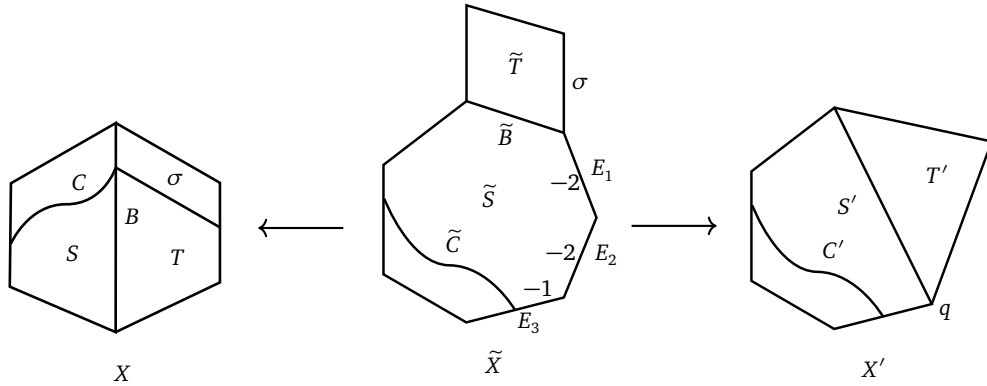

FIGURE 5. The central fiber X is replaced by X' in a type 2 flip.

Construct (X', D') from (X, D) as follows (see Figure 5). Let $\widetilde{S} \to S$ be the blow up of S three times, first at p, second at the intersection of the exceptional divisor of the first blow-up with the proper transform of C, and third at the intersection point of the

exceptional divisor of the second blow up with the proper transform of C. Equivalently, $\widetilde{S}$ is the minimal resolution of the blow-up of S at the unique subscheme of C of length 3 supported at p. Denote by $\widetilde{C}$ and $\widetilde{B}$ the proper transforms of C and B in $\widetilde{S}$. Let $\widetilde{X}$ be the union of $\widetilde{S}$ and T, glued along $\widetilde{B} \subset \widetilde{S}$ and $B \subset T$ via the canonical isomorphism $\widetilde{B} \to B$ induced by the identity on B. Let E_i be the proper transform in $\widetilde{S}$ of the exceptional divisor of the ith blowup, for $i = 1, 2, 3$. Let S' be obtained from $\widetilde{S}$ by contracting $\widetilde{E}_1$ and $\widetilde{E}_2$. Let T' be obtained from T by contracting σ. Let X' be the union of S' and T' glued along the image of $\widetilde{B}$ in S' and the image of B in T' via the isomorphism between the two induced by the identity on B. Let $B' \subset X'$ be the image of either of these curves. Let $C' \subset X'$ be the image of $\widetilde{C}$, and set $D' = C'$. Let $\nu \subset X'$ be the image of $E_3 \subset \widetilde{S}$.

We would like to prove that we can replace (X, D) on the central fiber by (X', D') under an additional hypothesis on the structure of $(\mathcal{X}, \mathcal{D})$ along B. Assume that there exists a family of (not necessarily projective) curves $\mathcal{P} \to \Delta$, smooth over Δ°, and with a single node p on the central fiber, an open subset $\mathcal{U} \subset \mathcal{X}$ containing B, and an isomorphism $\mathcal{U} \cong B \times \mathcal{P}$ over Δ. Assume, furthermore, that the first projection $\mathcal{U} \to \mathcal{P}$ restricts to an isomorphism $\mathcal{D} \cap \mathcal{U} \to \mathcal{P}$.

Proposition 5.6. *Let $(\mathcal{X}, \mathcal{D}) \to \Delta$ be a family of log surfaces as described above. There exists a flat family $(\mathcal{X}', \mathcal{D}') \to \Delta$ isomorphic to $(\mathcal{X}, \mathcal{D})$ over Δ° such that the central fiber of $(\mathcal{X}', \mathcal{D}') \to \Delta$ is (X', D'). Furthermore, $\mathcal{X}'$ is $\mathbb{Q}$-factorial and has canonical singularities.*

Remark 5.7. Note that (X', D') is log canonical. Also note that it depends only on (X, D), not on the family $(\mathcal{X}, \mathcal{D}) \to \Delta$.

Before proving Proposition 5.6, we look at X' and its two components S' and T' in more detail. The contraction $\widetilde{S} \to S'$ results an $A_2 = \frac{1}{3}(1, 2)$ singularity on S' at the image point of the chain E_1, E_2. The contraction $T \to T'$ results in a $\frac{1}{3}(1, 1)$ singularity at the image point of the curve σ. These two singularities are glued together in X', say

at a point q. The complete local ring of X' at q is the ring of invariants of

$$\mathbb{C}[x, y, z]/(xy)$$

under the action of μ_3 where an element $\zeta \in \mu_3$ acts by

$$\zeta \cdot (x, y, z) \mapsto \zeta(x, \zeta^2 y, \zeta z).$$

Finally, observe that the Picard ranks of the new surfaces are given by

$$\rho(S') = \rho(S) + 1, \text{ and}$$

$$\rho(T') = \rho(T) - 1.$$

On S', we have the intersection numbers

$$(C')^2 = C^2 - 3,$$

$$v^2 = -\frac{1}{3}, \text{ and}$$

$$B' \cdot v = \frac{1}{3}.$$

On S' and T', we have the following intersection numbers of B'

$$\left(B'|_{S'}\right)^2 = (B|_S)^2 - \frac{1}{3}, \text{ and}$$

$$\left(B'|_{T'}\right)^2 = (B|_T)^2 + \frac{1}{3}.$$

5.2.1. *Proof of Proposition 5.6: The non-singular case.* Assume that $\mathcal{P}$ is non-singular. Then both $\mathcal{X}$ and $\mathcal{D}$ are non-singular.

We construct $\mathcal{X}'$ from $\mathcal{X}$ by an explicit sequence of blow ups and blow downs. We denote the intermediate steps in this process by $\mathcal{X}^{(i)}$. Throughout, $\mathcal{D}^{(i)} \subset \mathcal{X}^{(i)}$ denotes the closure of $\mathcal{D}\big|_{\Delta^\circ}$ in $\mathcal{X}^{(i)}$, or equivalently the proper transform of $\mathcal{D}$ in $\mathcal{X}^{(i)}$. Let $\pi \colon \mathcal{X} \to \mathcal{Y}$ be the contraction of σ. All the $\mathcal{X}^{(i)}$ will be projective over $\mathcal{Y}$.

The first three steps consist of blow-ups; their central fibers are depicted in Figure 6.
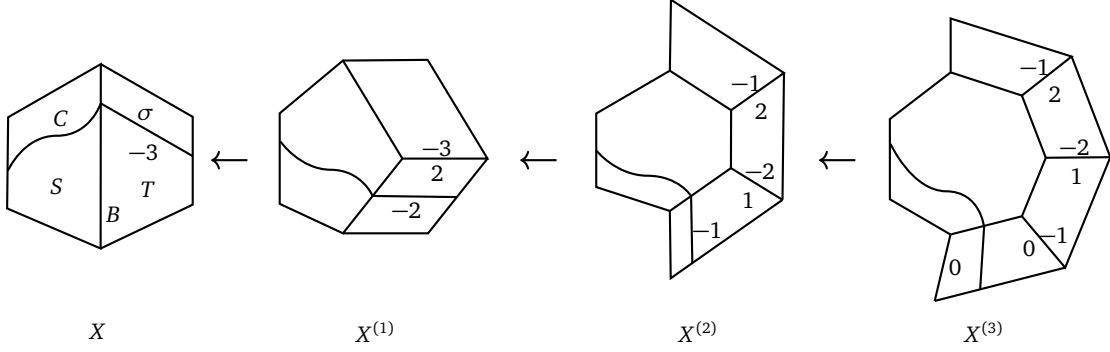
FIGURE 6. The central fibers $X^{(i)}$ of the first three blow ups $\mathcal{X}^{(i)}$ of $\mathcal{X}$.

The first step $\mathcal{X}^{(1)} \to \mathcal{X}$ is the blow up at σ. Let $E^{(1)} \subset \mathcal{X}^{(1)}$ be the exceptional divisor. Note that the central fiber of $\mathcal{X}^{(1)} \to \Delta$ is the union of $E^{(1)}$ and the proper transform $X^{(1)}$ of X. The surface $X^{(1)}$ has two smooth irreducible components, namely $\mathrm{Bl}_p\, S$ and T, which intersect transversely along the proper transform of B. Set $\sigma^{(1)} = E^{(1)} \cap \mathcal{D}^{(1)}$.

The following lemma identifies the normal bundle of σ and hence the isomorphism class of $E^{(1)}$.

Lemma 5.8. *The normal bundle $N_{\sigma/\mathcal{X}}$ is given by*

$$N_{\sigma/\mathcal{X}} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-3).$$

As a result, we have $E^{(1)} \cong \mathbb{F}_2$, and $\sigma^{(1)}$ is the unique -2 curve on $E^{(1)}$.

Proof. We have the exact sequence of bundles

$$0 \to N_{\sigma/T} \to N_{\sigma/\mathcal{X}} \to N_{T/\mathcal{X}}\big|_\sigma \to 0,$$

in which the kernel is $\mathcal{O}(-3)$ and the cokernel is $\mathcal{O}(-1)$. Therefore, the only possibilities for $N_{\sigma/\mathcal{X}}$ are $\mathcal{O}(-i) \oplus \mathcal{O}(-4+i)$ for $i = 1, 2$.

A simple divisor class computation shows that $\mathcal{D}^{(1)} \cap E$ is an effective divisor on E of self-intersection (-2). The map $\mathcal{D}^{(1)} \to \mathcal{D}$ is the blow-up of $\mathcal{D}$ along σ. Since $\mathcal{D}$ is non-singular, this is an isomorphism. Therefore, the scheme-theoretic intersection $\mathcal{D}^{(1)} \cap E$ is a section of $E \to \sigma$. Among the two possibilities for E given by $i = 1, 2$, only $i = 1$ yields a surface with a section of self-intersection (-2). The result follows. $\qquad\square$

The second step $\mathcal{X}^{(2)} \to \mathcal{X}^{(1)}$ is the blow up of $\mathcal{X}^{(1)}$ along $\sigma^{(1)}$. Define $E^{(2)}$, $\mathcal{D}^{(2)}$, and $\sigma^{(2)}$ as before. By similar computation as in the proof of Lemma 5.8, we get that $E^{(2)} \cong \mathbb{F}_1$ and $\sigma^{(2)} \subset E^{(2)}$ is the unique curve of self-intersection (-1).

The third step $\mathcal{X}^{(3)} \to \mathcal{X}^{(2)}$ is the blow up of $\mathcal{X}^{(3)}$ along $\sigma^{(2)}$. Define $E^{(3)}$, $\mathcal{D}^{(3)}$, and $\sigma^{(3)}$ as before. Again, by a similar computation as before, we get that $E^{(3)} \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\sigma^{(3)} \subset E^{(3)}$ is a line of a ruling, opposite to the fibers of $E^{(3)} \to \sigma^{(2)}$.

The next three steps consist of divisorial contractions.

Let $\mathcal{X}^{(3)} \to \mathcal{X}^{(4)}$ be the contraction in which the lines of the ruling of $\sigma^{(3)}$ are contracted. This results in the contraction of $E^{(3)}$ in the opposite direction as compared with the contraction in $\mathcal{X}^{(3)} \to \mathcal{X}^{(2)}$. Note that this is the contraction of the $K_{\mathcal{X}^{(3)}}$-negative extremal ray of $\mathrm{NE}(\pi)$ spanned by $\sigma^{(3)}$, and thus its existence is guaranteed by Theorem 2.12. Since this is a divisorial contraction, $\mathcal{X}^{(4)}$ is $\mathbb{Q}$-factorial with canonical singularities. In fact, it turns out that the contraction does not introduce any new singularities (by Theorem 2.13(1)). Let $\mathcal{D}^{(4)} \subset \mathcal{X}^{(4)}$ be the image of $\mathcal{D}^{(3)}$. The images of $E^{(1)}$ and $E^{(2)}$ in $\mathcal{X}^{(4)}$ lie away from $\mathcal{D}^{(4)}$. The image $\overline{E}^{(2)}$ of $E^{(2)}$ is isomorphic to $\mathbb{P}^2$. The image of is isomorphic to $E^{(1)}$; we denote it by the same notation.

Let $\mathcal{X}^{(4)} \to \mathcal{X}^{(5)}$ be the map that contracts $\overline{E}^{(2)}$ to a point. This is the contraction of the $K_{\mathcal{X}^{(4)}}$-negative extremal ray of $\mathrm{NE}(\pi)$ spanned by a line in $\overline{E}^{(2)}$. Again, $\mathcal{X}^{(5)}$ is $\mathbb{Q}$-factorial with canonical singularities. The image $\overline{E}^{(1)}$ of $E^{(1)}$ is a surface of Picard rank 1; the only curve class on it is $[\tau]$, where τ is the image of $E^{(1)} \cap \mathrm{Bl}_p S$. The only new singularity on

$\mathfrak{X}^{(5)}$ is at the image point of $\overline{E}^{(2)}$; it is the quotient singularity $\mathbb{A}^3/\mathbb{Z}_2$ where the generator of $\mathbb{Z}_2$ acts by $(x, y, z) \mapsto (-x, -y, -z)$ (by Theorem 2.13(5)).

Finally, let $\mathfrak{X}^{(5)} \to \mathfrak{X}'$ be the contraction that maps $\overline{E}^{(1)}$ to a point. It is the contraction of the $K_{\mathfrak{X}^{(5)}}$-negative extremal curve $[\tau]$ of $\mathrm{NE}(\pi)$.

Set $X' = \mathfrak{X}'_0$ and $D' = \mathcal{D}'_0$. Observe that the transformation from X to X' is exactly as described in Proposition 5.6—on one component S, it is the result of two blow ups on C followed by the contraction of a $(-2, -2)$ chain of $\mathbb{P}^1$s, resulting in an A_2 singularity. On the other component T, it is just the contraction of σ, resulting in a $\frac{1}{3}(1, 1)$ singularity. The proof of Proposition 5.6 is now complete, under the assumption that $\mathcal{P}$ is non-singular.

5.2.2. *Proof of Proposition 5.6: The general case.* We now boot-strap to the general case from the non-singular case.

Since the central fiber of $\mathcal{P} \to \Delta$ has a nodal singularity at p, the surface $\mathcal{P}$ has an A_n singularity at p for some $n \geq 0$. We have already dispensed the case $n = 0$, so assume $n \geq 1$. Let $\mathcal{P}^{(0)} \to \mathcal{P}$ be the minimal resolution of singularities. The exceptional divisor of $\mathcal{P}^{(0)}_0$ consists of a chain of n rational curves. Set

$$\mathfrak{X}^{(0)} = \left(\mathcal{U} \times_{\mathcal{P}} \mathcal{P}^{(0)}\right) \bigcup_{\mathcal{U} \setminus B} (\mathfrak{X} \setminus B)$$

$$= \left(B \times \mathcal{P}^{(0)}\right) \bigcup_{\mathcal{U} \setminus B} (\mathfrak{X} \setminus B).$$

Then we have a map $\mathfrak{X}^{(0)} \to \mathfrak{X}$, which is a resolution of singularities. The central fiber $X^{(0)}$ of $\mathfrak{X}^{(0)} \to \Delta$ is the union

$$X^{(0)} = S^{(0)} \cup E_1^{(0)} \cup \cdots \cup E_n^{(0)} \cup T^{(0)},$$

where $S^{(0)}$ and $T^{(0)}$ denote the strict transforms of S and T, and each $E_i^{(0)}$ is isomorphic to $B \times \mathbb{P}^1$. Let $\mathcal{D}^{(0)}$ be the proper transform of $\mathcal{D}$, and $\sigma^{(0)}$ and $C^{(0)}$ the proper transforms of σ and C, respectively. Since the map $\mathcal{D} \to \mathcal{P}$ is an isomorphism over an open subset

of $\mathcal{P}$ containing p, the map $\mathcal{D}^{(0)} \cap \mathcal{U} \to \mathcal{P}^{(0)}$ is an isomorphism over an open subset of $\mathcal{P}^{(0)}$ containing the pre-image of p in $\mathcal{P}^{(0)}$. In particular, $\mathcal{D}^{(0)}$ is non-singular. Also, the intersection $D_i := \mathcal{D}^{(0)} \cap E_i^{(0)}$ is a section of $E_i^{(0)} \to \mathbb{P}^1$. See Figure 7 for a picture of $(\mathcal{X}^0, \mathcal{D}^{(0)})$.
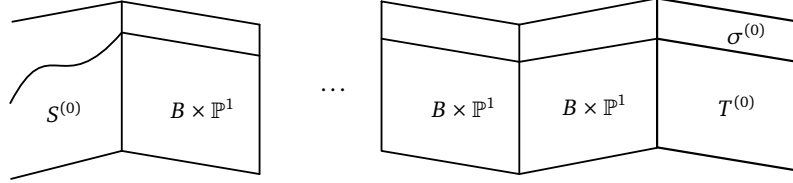
FIGURE 7. The accordion-like central fiber of $(\mathcal{X}^{(0)}, \mathcal{D}^{(0)})$

We now apply the non-singular case of Proposition 5.6 repeatedly to pair $\left(\mathcal{X}^{(0)}, \mathcal{D}^{(0)}\right)$.

First flip the (-3) curve $\sigma^{(0)}$ by applying Proposition 5.6 to an open subset of $\mathcal{X}^{(0)}$ containing $\sigma^{(0)}$. The role of S and T is played by $E_n^{(0)} = B \times \mathbb{P}^1$ and $T^{(0)}$, respectively. The resulting threefold $\mathcal{X}^{(1)}$ (see Figure 8) has central fiber

$$X^{(1)} = S^{(1)} \cup E_1^{(1)} \cup \cdots \cup E_n^{(1)} \cup T^{(1)}$$

as shown in Figure 8, where $S^{(1)} = S^{(0)}$ and $E_i^{(1)} = E_i^{(0)}$ for $i = 1, \ldots, n-1$, whereas $E_n^{(1)}$ is a surface with an A_2 singularity obtained by three blow ups and two blow downs from $E_n^{(0)}$, and $T^{(1)}$ is obtained from $T^{(0)}$ by contracting $\sigma^{(0)}$. Note that the transformations $E_n^{(0)} \dashrightarrow E_n^{(1)}$ and $T^{(0)} \to T^{(1)}$ are simply the transformations $S \dashrightarrow S'$ and $T \to T'$ from Proposition 5.6. The proper transform of D_n on $E_n^{(0)}$ is a (-3) curve $\sigma^{(1)}$ on $E_n^{(1)}$. Note that $\sigma^{(1)}$ lies in the non-singular locus of $E_n^{(1)}$ and $\mathcal{X}^{(1)}$, away from $T^{(1)}$.
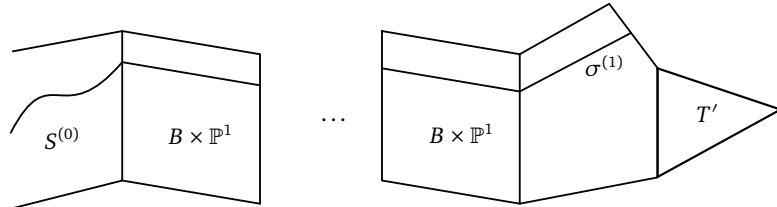
FIGURE 8. A modified accordion after a (-3) flip

Once more, flip the (-3) curve $\sigma^{(1)}$ by applying Proposition 5.6 to an open subset of $\mathcal{X}^{(1)}$ containing $\sigma^{(1)}$. Now the role of S and T is played by $E_{n-1}^{(1)}$ and $E_n^{(1)}$, respectively. The resulting threefold $\mathcal{X}^{(2)}$ has central fiber

$$X^{(2)} = S^{(2)} \cup E_1^{(2)} \cup \cdots \cup E_{n-1}^{(2)} \cup E_n^{(2)} \cup T^{(2)},$$

where the only components that are different from their previous counterparts are $E_{n-1}^{(2)}$ and $E_n^{(2)}$. The surface $E_{n-1}^{(2)}$ is obtained by three blow ups and two blow downs from $E_{n-1}^{(1)}$, and $E_n^{(2)}$ is obtained from $E_n^{(1)}$ by contracting $\sigma^{(1)}$. The proper transform of D_{n-1} on $E_{n-1}^{(1)}$ is a (-3) curve $\sigma^{(2)}$ on $E_{n-1}^{(2)}$, which lies in the non-singular locus of $E_{n-1}^{(2)}$ and $\mathcal{X}^{(2)}$, and away from $E_n^{(2)}$.

Continue flipping the (-3) curves $\sigma^{(i)}$, for $i = 2, 3, \ldots, n$, resulting in a threefold $\mathcal{X}^{(n+1)}$ which has central fiber

$$X^{(n+1)} = S^{(n+1)} \cup E_1^{(n+1)} \cup \cdots \cup E_n^{(n+1)} \cup T^{(n+1)}.$$

Note that we now have $S^{(n+1)} \cong S'$, obtained by three blow ups and two blow downs from S as described in Proposition 5.6, and $T^{(n+1)} \cong T'$, obtained by contracting the (-3) curve σ on T. The intermediate components $E_i^{(n+1)}$ are obtained by 3 blow-ups and 3 blow-downs on $E_i^{(0)} \cong B \times \mathbb{P}^1$. Notice that the curves that are blown down are contracted under the map to B. As a result, the projection map $E_i^{(0)} \to B$ survives as a regular map $E_i^{(n+1)} \to B$.

Note that $K_{X^{(n+1)}} + wD^{(n+1)}$ is nef but not ample on $E_i^{(n+1)}$; it is trivial on the fibers of $E_i^{(n+1)} \to B$. Recall that $\mathcal{X} \to \mathcal{Y}$ is the contraction of σ, and we have a projective morphism $\psi \colon \mathcal{X}^{(n+1)} \to \mathcal{Y}$. The bundle $K_\psi + w\mathcal{D}$ is nef, hence semi-ample by Theorem 2.20. It gives a divisorial contraction $\mathcal{X}^{(n+1)} \to \mathcal{X}'$ in which all $E_i^{(n+1)}$ are contracted to B.

Let $\mathcal{D}'$ be the image of $\mathcal{D}^{(n+1)}$. By construction $\mathcal{X}'$ is $\mathbb{Q}$-factorial with canonical singularities. Furthermore, the central fiber (X', D') is as required in Proposition 5.6. The proof of Proposition 5.6 is now complete in general.

6. STABLE REPLACEMENTS OF UNSTABLE PAIRS

The goal of this section is to prove properness of the moduli stack of stable log quadrics $\mathfrak{X}$ by enhancing the partial valuative criterion of properness Proposition 3.11. The key step is to construct all limits of stable log quadrics over a punctured DVR and verify that the limits are indeed stable log quadrics.

Let Δ be a DVR and $(\mathfrak{X}_\eta, \mathcal{D}_\eta) \cong ((\mathbb{P}^1 \times \mathbb{P}^1)_\eta, C)$ be a stable log surface over the generic point η of Δ where C is a smooth curve of bi-degree $(3,3)$. Possibly after a finite base change, $(\mathfrak{X}_\eta, \mathcal{D}_\eta)$ extends to a family $(\overline{\mathfrak{X}}, \overline{\mathcal{D}}) \to \Delta$ such that the central fiber $(\overline{X}, \overline{D})$ is a stable log surface and both $K_{\overline{\mathfrak{X}}/\Delta}$ and $\overline{\mathcal{D}}$ are $\mathbb{Q}$-Cartier by Proposition 3.11. We describe all possible $(\overline{X}, \overline{D})$ in subsequent subsections and verify that they all satisfy the index condition Definition 3.4. This confirms that $(\overline{\mathfrak{X}}, \overline{\mathcal{D}}) \in \mathfrak{X}(\Delta)$, and shows the valuative criteria of properness of $\mathfrak{X}$. We do this explicitly and independently of the proof sketched after Proposition 3.11.

Consider $\phi \colon \mathcal{D}_\eta \to \mathbb{P}^1_\eta$ induced by the first projection $(\mathbb{P}^1 \times \mathbb{P}^1)_\eta \to \mathbb{P}^1_\eta$ as a η-valued point of $\mathcal{H}^3_4$. Let $\mathcal{C} \to \mathcal{P}$ be its unique extension to a Δ-valued point of $\overline{\mathcal{H}}^3_4(1/6 + \epsilon)$, possibly after a base-change. Note that $\mathcal{P} \to \Delta$ is an orbi-nodal curve of genus 0. Let $\mathcal{E}$ be the Tschirnhausen bundle of $\phi \colon \mathcal{C} \to \mathcal{P}$. By the procedure described in Section 4, ϕ gives a divisor $D(\phi)$ in $\mathbb{P}\mathcal{E}$. By a closer inspection, we can see that $(\mathfrak{X}, (2/3 + \epsilon)\mathcal{D})$ is klt (by modifying proof of Proposition 4.2) Let $(\mathfrak{X}, \mathcal{D})$ be the coarse space of $(\mathbb{P}\mathcal{E}, \ell(\phi))$. Let (X, D) be the fiber of $(\mathfrak{X}, \mathcal{D})$ over the closed point $0 \in \Delta$. By Proposition 4.6, the fibers of $(\mathfrak{X}, \mathcal{D}) \to \Delta$ are semi-stable log quadric surfaces. By construction, the general fiber is also stable, but the special fiber need not be. Since $\mathfrak{X}$ is $\mathbb{Q}$-factorial and the central fiber (X, D) satisfies the index condition by Remark 4.4, the family $(\mathfrak{X}, \mathcal{D}) \to \Delta$ is $\mathbb{Q}$-Gorenstein Lemma 3.9. The goal of this section is to prove the following.

Theorem 6.1 (Stabilization)**.** *Let $(\mathfrak{X}, \mathcal{D}) \to \Delta$ be as above. There exists a $\mathbb{Q}$-Gorenstein family $(\overline{\mathfrak{X}}, \overline{\mathcal{D}}) \to \Delta$ of stable log quadrics with generic fiber $(\mathfrak{X}_\eta, \mathcal{D}_\eta)$ on η. Furthermore,*

the central fiber $(\overline{X}, \overline{D})$ of $(\overline{\mathcal{X}}, \overline{\mathcal{D}}) \to \Delta$ depends only on the central fiber (X, D) of the original family $(\mathcal{X}, \mathcal{D}) \to \Delta$.

Since $\mathfrak{X}$ is separated, the family $(\overline{\mathcal{X}}, \overline{\mathcal{D}}) \to \Delta$ is unique up to isomorphism. Theorem 6.1 proves the valuative criteria for properness for $\mathfrak{X}$. We highlight that, after obtaining the semi-stable family $(\mathcal{X}, \mathcal{D})$, a further base change is not necessary to get to the stable family. Furthermore, the central fiber of the stable family depends only on the central fiber of the original family.

Outline of proof of Theorem 6.1. If $K_X + (2/3 + \epsilon)D$ is ample for some $\epsilon > 0$, then $(\overline{\mathcal{X}}, \overline{\mathcal{D}}) = (\mathcal{X}, \mathcal{D})$, and there is nothing to prove. The end of Section 4 lists the possibilities for $C \to P$ for which $K_X + (2/3 + \epsilon)D$ fails to be ample for all $\epsilon > 0$. In all these cases, we construct $(\overline{\mathcal{X}}, \overline{\mathcal{D}})$ from $(\mathcal{X}, \mathcal{D})$ by explicitly running a minimal model program on the threefold $\mathcal{X}$ (see § 2.2) using the birational transformations described in Section 5. This program consists of the following two steps.

Step 1 (Flips): By a sequence of flips on the central fiber of $\mathcal{X}$, we construct $(\mathcal{X}', \mathcal{D}') \to \Delta$ with slc fibers and $\mathbb{Q}$-factorial total space $\mathcal{X}'$ such that $K_{\mathcal{X}'} + (2/3 + \epsilon)\mathcal{D}'$ is $\mathbb{Q}$-Cartier and nef for all sufficiently small $\epsilon > 0$. Our construction shows that the central fiber of $(\mathcal{X}', \mathcal{D}')$ depends only on the central fiber of $(\mathcal{X}, \mathcal{D})$.

Step 2 (Contractions): Set $w = 2/3 + \epsilon$, where $\epsilon > 0$ is such that $K_{\mathcal{X}'} + (2/3 + \epsilon)\mathcal{D}'$ is nef. By Theorem 2.20, the divisor $K_{\mathcal{X}'} + w\mathcal{D}'$ is semi-ample. We set

$$\overline{\mathcal{X}} = \mathrm{Proj}\left(\bigoplus_{n \geq 0} H^0\left(\mathcal{X}', n\left(K_{\mathcal{X}'} + w\mathcal{D}'\right)\right) \right),$$

and let $\overline{\mathcal{D}}$ be the image of $\mathcal{D}'$ in $\overline{\mathcal{X}}$. For this step, it is clear that the central fiber $(\overline{X}, \overline{D})$ of $(\overline{\mathcal{X}}, \overline{\mathcal{D}}) \to \Delta$ depends only on the central fiber of $(\mathcal{X}', \mathcal{D}')$. We describe $(\overline{X}, \overline{D})$ explicitly, culminating in the classification in Table 1. It is easy to check from the description that $(\overline{X}, w\overline{D})$ is slc. We also observe that $\overline{D} \subset \overline{X}$ is a Cartier divisor that stays away from the non-Gorenstein singularities of $\overline{X}$. Hence, $(\overline{X}, \overline{D})$ satisfies the index condition.

Furthermore, by construction, both $K_{\overline{\mathcal{X}}}$ and $\overline{\mathcal{D}}$ are $\mathbb{Q}$-Cartier divisors, so the family $(\overline{\mathcal{X}}, \overline{\mathcal{D}}) \to \Delta$ is $\mathbb{Q}$-Gorenstein by Lemma 3.9.

To complete the proof of Theorem 6.1, we must carry out the two steps in each case listed at the end of Section 4. We do this in separate subsections that follow. □

Remark 6.2. In all the cases, it is possible to show directly that $K_{\mathcal{X}'} + w\mathcal{D}'$ is semi-ample, avoiding appealing to the log abundance theorem. In fact, our proof that $K_{\mathcal{X}'} + w\mathcal{D}'$ is nef also yield it is semi-ample on the central fiber. To deduce that it is semi-ample on the whole threefold, it suffices to show that $H^i(X', n(K_{X'} + wD')) = 0$ for $i > 0$ and for sufficiently large and divisible n. Proving this vanishing is also fairly easy from the geometry of (X', D'). Nevertheless, we appeal to the log abundance theorem to keep the length of the proof reasonable.

6.1. **Maroni special covers.** Suppose $C \to P$ is as in case (1) of the unstable list from page 45. That is, $P \cong \mathbb{P}^1$ and $C \to P$ is Maroni special. In this case, $X \cong \mathbb{F}_2$ and $D \subset X$ is a divisor of class $3\sigma + 6F$.

Step 1 (Flips): In this case, $K_{\mathcal{X}} + w\mathcal{D}$ is already nef, so we do not need any flips.

Step 2 (Contractions): The only $(K_X + wD)$-trivial curve is σ. The contraction step contracts $\sigma \subset X$ to a point, resulting in $\overline{X}$ isomorphic to the weighted projective plane $\mathbb{P}(1,1,2)$.

There are two possibilities on how the curve $\overline{D}$ interacts with the unique singular point $p \in \overline{X}$. The first possibility is that $D \subset X$ is disjoint from σ. In this case, $\overline{D}$ is away from the singularity. The second possibility is that $D \subset X$ contains σ as a component. In this case $D = \sigma \cup E$, where E does not contain σ and $E \cdot \sigma = 2$. Therefore, $\overline{D} = \overline{E}$; this passes through the singularity of X and has either a node or a cusp there, depending on whether E intersects σ transversely at 2 points or tangentially at 1 point. The two steps in required for the proof of Theorem 6.1 are thus complete.

6.2. **Hyperelliptic covers.** Suppose $C \to P$ is as in case (2) of the unstable list from page 45. That is, $P \cong \mathbb{P}^1$ and $C = \mathbb{P}^1 \cup H$, where H is a hyperelliptic curve of genus 4 attached nodally to $\mathbb{P}^1$ at one point. In this case, $S \cong \mathbb{F}_4$ and D is the union of $\sigma \cong \mathbb{P}^1$ and a divisor of class $2\sigma + 9F$ isomorphic to H; we denote the divisor also by the letter H. Note that H intersects σ at a unique point, say p.

Step 1 (Flips): Let $(\mathcal{X}', \mathcal{D}')$ be the family obtained from $(\mathcal{X}, \mathcal{D})$ by flipping the -4 curve σ; this flip is constructed in § 5.1. Let (X', D') be the central fiber of $(\mathcal{X}', \mathcal{D}') \to \Delta$. Recall that the relationship between X and X' is given by the diagram

(6.1)
$$X \leftarrow \widetilde{X} \to X'$$

where $\widetilde{X}$ is obtained from X by blowing up the point p and the intersection point q of the proper transform of H and the exceptional divisor of the first blowup. Let F be the fiber of $X \to \mathbb{P}^1$ through p. Denote the proper transforms of σ, F, and H by the same letters, and denote by E_1 and E_2 be the exceptional divisors of the two blow-ups. Then $\widetilde{X} \to X'$ is obtained by contracting σ and E_1.

Step 2 (Contractions): There are three possibilities for the ramification behavior of $H \to \mathbb{P}^1$ at p, which dictate the result of the contraction step. To analyze the contracted curves, it is necessary to look at the configuration of the curves $\{\sigma, H, F, E_1, E_2\}$ on $\widetilde{X}$, which we encode by its dual graph.

Case 1: $H \to \mathbb{P}^1$ is unramified at p.

In this case, $K_{\mathcal{X}'} + w\mathcal{D}'$ is ample, and hence $(\overline{\mathcal{X}}, \overline{\mathcal{D}}) = (\mathcal{X}', \mathcal{D}')$. To see the ampleness, observe that on $\widetilde{X}$ we have the dual graph

.
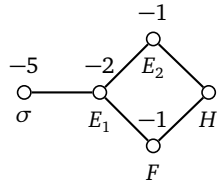
Let E_2' and F' be the image in X' of E_2 and F on $\widetilde{X}$. From the dual graph above, we obtain the following intersection table on X'

	E_2'	F'	$K_{X'}$	D'
E_2'	$-4/9$	$5/9$	$-2/3$	1
F'	$5/9$	$-4/9$	$-2/3$	$1.$

Since X' is a $\mathbb{Q}$-factorial surface of Picard rank 2, and E_2' and F' have negative self-intersection, they must span $\overline{\mathrm{NE}}(X')$. We have

$$(K_{X'} + wD') \cdot E_2' = (K_{X'} + wD') \cdot F' = \epsilon > 0,$$

so $K_{X'} + w\mathcal{D}'$ is ample.

Note that the surface $\overline{X} = X'$ has a $\frac{1}{9}(1,2)$ singularity obtained by contracting the chain (σ, E_1). The divisor $\overline{D} = D'$ stays away from the singularity.

Case 2: $H \to \mathbb{P}^1$ is ramified at p.

In this case, the dual graph is
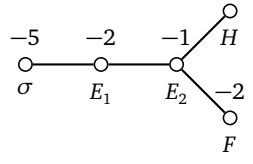
As in case 1, we get that $\overline{\mathrm{NE}}(X')$ is spanned by the images E_2' and F' of E_2 and F, and we have

$$(K_{X'} + wD') \cdot E_2' = \epsilon > 0, \text{ and}$$

$$(K_{X'} + wD') \cdot F' = 0.$$

Therefore, the contraction step contracts F', resulting in an A_1 singularity on $\overline{X}$. The divisor $\overline{D}$ stays away from the singularity.

Remark 6.3. Similarly to Remark 6.6, when H is smooth, $\overline{X}$ is determined from a hyperelliptic curve H with a hyperelliptic divisor $2p$, and H is on non-singular locus of $\overline{X}$.

Case 3: H contains F as a component.

6.2.1. *H contains F as a component.* In this case, let $H = F \cup G$, where G is the residual curve. We have the dual graph

For the same reason as in case 2, $K_{X'} + wD'$ is nef and it contracts the curve F, resulting in a surface $\overline{X}$ with an A_1 singularity. Note, however, that the divisor $\overline{D}$–which is the image of G–passes through the A_1 singularity. If F intersects G transversely in 2 distinct points, then $\overline{D}$ has a node at the A_1 singularity. If F intersects G tangentially at 1 point, then $\overline{D}$ has a cusp at the A_1 singularity.

The two steps in required for the proof of Theorem 6.1 are now complete.

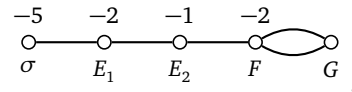*Remark* 6.4. We record some properties of $\overline{X}$ and $\overline{D}$ obtained in each case. In case 1, $\overline{X}$ is obtained from X by two blow-ups and and two blow-downs. The blow-ups use the auxiliary data of the point p on the hyperelliptic component H of D and the tangent direction to H at p; the blow-downs do not require any auxiliary data. The automorphism group of X acts transitively on the necessary auxiliary data, and therefore the isomorphism class of $\overline{X}$ is independent of D. The blow-downs result in a unique singular point on $\overline{X}$ corresponding to a $\frac{1}{9}(1,2)$ singularity. It is easy to see that $\overline{X}$ is not a toric surface. In case 2 and case 3, the tangent direction to H at p is along the fiber of X through p. As a result, $\overline{X}$ is a toric surface. More precisely, it is easy to figure out that $\overline{X}$ is isomorphic to $\mathbb{P}(1,2,9)$.

Remark 6.5. Observe that the covers $C \to P$ in cases 2 and 3 are specializations of the covers in case 1. By considering the family of surfaces $\overline{X}$ in such a specialization, we see that the non-toric surface $\overline{X}$ in case 1 specializes to $\mathbb{P}(1,2,9)$. In other words, $\overline{X}$ is a smoothing of the A_1 singularity on $\mathbb{P}(1,2,9)$. We can check that in this family of surfaces, both K and $\mathcal{D}$ are $\mathbb{Q}$-Gorenstein, so the family is a $\mathbb{Q}$-Gorenstein family (Lemma 3.9).

Remark 6.6. Suppose we are in the generic case, namely with H smooth and $H \to \mathbb{P}^1$ unramified at p. The hyperelliptic involution of H extends to an automorphism of $\widetilde{X}$. This automorphism fixes σ point-wise, fixes E_1 as a set, and interchanges E_2 and F. It descends to an automorphism on $\overline{X}$ that interchanges the two extremal rays $\overline{E}_1$ and $\overline{F}$ of the $\overline{\mathrm{NE}}(\overline{X})$.

6.3. **The $\mathbb{F}_3 - \mathbb{F}_3$ case.** Suppose we are in case (3a) of the unstable list from page 45. That is, $C = C_1 \cup C_2$ mapping to $P = \mathbb{P}^1 \cup \mathbb{P}^1$, where C_i is the disjoint union of $\mathbb{P}^1$ and a hyperelliptic curve H_i of genus 2. In this case, $X = X_1 \cup X_2$, where $X_i \cong \mathbb{F}_3$ and $D_i \subset X_i$ is the disjoint union of the directrix σ_i and a curve H_i of class $2\sigma_i + 6F$. Since C is connected, we note that σ_1 intersects X_2 and is disjoint from σ_2, and vice-versa.

Step 1 (Flips): Let $(\mathcal{X}', \mathcal{D}')$ be the family obtained from $(\mathcal{X}, \mathcal{D})$ by flipping the -3 curves σ_1 and σ_2. Let (X', D') be the central fiber of $(\mathcal{X}' \to \mathcal{D}') \to \Delta$. The surface X' is the union of two components $X_1' \cup X_2'$, where each X_i' is related to X_i by a diagram

$$X_i \leftarrow \widetilde{X}_i \to X_i'.$$

This diagram is given by Figure 5; the role of S and T is played by X_1 and X_2 while flipping σ_2 and by X_2 and X_1 while flipping σ_1. To recall, $\widetilde{X}_i \to X_i$ is the blow-up of X_i three times, first at $D_i \cap \sigma_i$, and two more times at the proper transform of D_i and the most recent exceptional divisor. Denote the exceptional divisor of the jth blowup by E_{ij} for $j = 1, 2, 3$; use the same letters to denote proper transforms; and denote by F the curve $X_1 \cap X_2$. Then, $\widetilde{X}_i \to X_i'$ is the blow down of E_{i1}, E_{i2}, and σ_i. Note that X_i' has

a μ_3 singularity at the image point of σ_i, and an A_2 singularity at the image point of $E_1 \cup E_2$; it is smooth elsewhere.

Step 2 (Contractions): We claim that $K_{\mathcal{X}'} + w\mathcal{D}'$ is already ample, and hence no contractions are necessary. In other words, we have $(\overline{\mathcal{X}}, \overline{\mathcal{D}}) = (\mathcal{X}', \mathcal{D}')$.

To show the ampleness, we must show that $K_{X'} + wD'$ is positive on $\overline{NE}(X'_i)$ for $i = 1, 2$. The dual graph of the configuration of curves $\{\sigma_i, E_1, E_2, E_3, H_i\}$ on $\widetilde{X}_i$ is
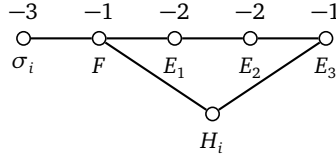
Denote by F' and E'_3 the images in X'_i of F and E_3, respectively. Using the dual graph above, we get the following intersection table on X'_i:

	E'_3	F'	K	D'
E'_3	$-1/3$	$1/3$	-1	1
F'	$1/3$	$-5/6$	$1/3$	1

Since X'_i is of Picard rank 2, and the two curves F' and E'_3 have negative self-intersection, they generate $\overline{NE}(X'_i)$. Now we compute

$$(K_{X'} + wD') \cdot E'_3 = \epsilon > 0$$

$$(K_{X'} + wD') \cdot F' = 1/6 + \epsilon > 0.$$

Hence $K_{X'} + wD'$ is ample on X_i for $i = 1, 2$.

The two steps required in the proof of Theorem 6.1 are now complete.

Remark 6.7. We record some properties of the $(\overline{X}, \overline{D})$ we found above.

First, note that $\overline{X}$ is determined from X by two length 3 subschemes of X_1 and X_2, namely the curvilinear subschemes of length 3 on H_1 and H_2 supported at $\sigma_1 \cap H_2$ and

$\sigma_2 \cap H_1$. All such data are equivalent modulo the action of the automorphism group of X. Therefore, the isomorphism type of $\overline{X}$ is uniquely determined.

Second, note that the two components of $\overline{X}$ are toric. To see this, note that there are toric structures on the surfaces X_i's such that the σ_i's and the curvilinear sub-schemes of length 3 above are torus fixed. Tracing through the transformation of X to $\overline{X}$, we see $\overline{X}$ may be represented as the a degenerate (non-normal) toric surface represented by the union of the quadrilaterals $\langle(-3,-2),(-3,-1),(3,1),(3,-2)\rangle$ and $\langle(-3,-1),(-3,2),(3,2),(3,1)\rangle$ (see Figure 9).
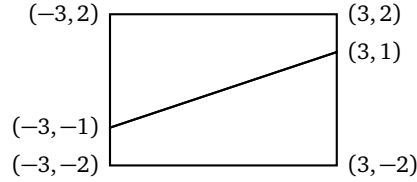
FIGURE 9. The non-normal toric surface $\overline{X}$ obtained in th $\mathbb{F}_3 - \mathbb{F}_3$ case

Finally, let $p, q \in \overline{X}$ be the images of σ_1, σ_2, respectively. Then $\overline{X}$ has the singularity type $(xy = 0) \subset \frac{1}{3}(1,2,1)$ at p, and $(xy = 0) \subset \frac{1}{3}(2,1,1)$ at q.

It turns out that $\overline{X}$ also appears as a stable limit in a different guise.

Proposition 6.8. $(\overline{X}, \overline{D})$ *is isomorphic to a log surface appearing in* (2) *of the stable list from page 44.*

Proof. Let $\overline{F} = \overline{X}_1 \cap \overline{X}_2$, and let $x_i \in H_i$ to be the point of X_i that gets blown up 3 times in the construction of $\overline{X}_i$ from X_i.

Let f be the class of in $\overline{X}_i$ of the image of the proper transform of a section τ of $X_i \cong \mathbb{F}_3$ which is triply tangent to H_i at x_i and satisfies $\tau^2 = 3$. Then we have

$$f^2 = 0, \quad f \cdot \overline{F} = 0, \text{ and } \overline{D}|_{\overline{X}_i} \cdot f = 3.$$

Moreover, there is a 1-parameter families of such sections τ, and the proper transforms of different sections yield disjoint images in $\overline{X}_i$. The section $\sigma_i + 3F$ is a particular such

τ. The image of its proper transform is the divisor $3\overline{F}$. Thus, the line bundle associated to f is base-point free. It induces a map

$$\pi_i \colon \overline{X}_i \to \mathbb{P}^1,$$

which is generically a $\mathbb{P}^1$-fibration.

Define the stack Y_i by

$$Y_i = \left[\operatorname{spec}\left(\bigoplus_{n \in \mathbb{Z}} \mathcal{O}_{\overline{X}_i}\left(n\overline{F}\right) \right) \middle/ \mathbb{G}_m \right].$$

The natural map $Y_i \to \overline{X}_i$ is the coarse space map, and the divisorial pullback of $\mathcal{O}_{\overline{X}_i}(\overline{F})$ to Y_i is Cartier. A simple local calculation shows that over the two singular points of $\overline{X}_i$, the map $Y_i \to \overline{X}_i$ has the form

$$\left[\operatorname{spec}\mathbb{K}[x, y] \middle/ \mu_3 \right] \to \operatorname{spec}\mathbb{K}[x, y]/\mu_3.$$

Let $0 \in \mathbb{P}^1$ be the image of $\overline{F} \subset \overline{X}_i$. Set $P_i = \mathbb{P}^1(\sqrt[3]{0})$. Since the scheme theoretic pre-image of 0 is $3\overline{F}$, which is 3 times a Cartier divisor on Y_i, the natural map $Y_i \to \mathbb{P}^1$ gives a map $\pi \colon Y_i \to P_i$. It is easy to check that $Y_i \to P_i$ is the $\mathbb{P}^1$-bundle

$$Y_i = \mathbb{P}(\mathcal{O}(5/3) \oplus \mathcal{O}(4/3)).$$

Note that $\overline{D}_i \subset \overline{X}_i$ lies away from the singularities of X_i. Hence, it gives a divisor on Y_i, which we denote by the same symbol. We also see that $\overline{D}_i \to P_i$ is of the divisor class $\mathcal{O}(3) \otimes \pi^* \mathcal{O}(-3)$, and therefore is obtained from the Tschirnhausen construction from a triple cover $C_i \to P_i$. Putting together the triple covers for $i = 1, 2$, we obtain an element $\phi \colon C_1 \cup C_2 \to P_1 \cup P_2$ of type (2) on page 44. $\qquad\square$

The description of $\overline{X}$ as the degenerate toric surface given by the subdivided polytope in Figure 9 can also be obtained using the alternate description of $\overline{X}$ as the coarse space of $\mathbb{P}(\mathcal{O}(4/3, 5/3) \oplus \mathcal{O}(5/3, 4/3))$ obtained in Proposition 6.8.

6.4. The $\mathbb{F}_1 - \mathbb{F}_1$ case. Suppose we are in case (3b) of the unstable list from page 46. That is, $C = C_1 \cup C_2$ mapping to $P = \mathbb{P}^1 \cup \mathbb{P}^1$, where C_i is a connected curve of genus 1; $X_i \cong \mathbb{F}_1$; and $D_i \subset X_i$ is a divisor of class $3\sigma_i + 3F$ intersecting the fiber $X_1 \cup X_2$ transversely.

Step 1 (Flips). In this case, we observe that $K_{\mathcal{X}'} + w\mathcal{D}'$ is nef. Its restriction to each $\mathbb{F}_1$ is a multiple of the class $\sigma + F$. Therefore, no flips are required; that is, $(\mathcal{X}', \mathcal{D}') = (\mathcal{X}, \mathcal{D})$.

Step 2 (Contractions). The only $K_{\mathcal{X}'} + w\mathcal{D}'$ trivial curves are the directrices σ_i's on the X_i'. Therefore, $\overline{X}$ is the union of two copies of $\mathbb{P}^2$ along a line, and $\overline{D}$ is $D_1 \cup D_2$. Each D_i is a cubic curve, intersecting the line of attachment transversely.

The two steps necessary for the proof of Theorem 6.1 are thus complete.

6.5. The $\mathbb{F}_3 - \mathbb{F}_1$ case. Suppose we are in case (3c) of the unstable list from page 46, which is a mixture of the two cases (3b) and (3a) treated before. That is, $C = C_1 \cup C_2$ mapping to $P = \mathbb{P}^1 \cup \mathbb{P}^1$, where C_1 is the disjoint union of $\mathbb{P}^1$ and a hyperelliptic curve of genus 2, and C_2 is a connected curve of genus 1. In this case $X = X_1 \cup X_2$ and $D = D_1 \cup D_2$, where $X_1 \cong \mathbb{F}_3$ and $X_2 \cong \mathbb{F}_1$; $D_1 \subset X_1$ is the disjoint union of the directrix σ_1 and a curve H_1 of class $2\sigma_1 + 6F$, and $D_2 \subset X_2$ is a divisor of class $3\sigma_2 + 3F$; both D_1 and D_2 intersect the fiber $X_1 \cap X_2$ transversely.

Step 1 (Flips): Let $(\mathcal{X}', \mathcal{D}')$ be obtained from $(\mathcal{X}, \mathcal{D})$ by flipping the -3 curve $\sigma_1 \subset X_1$. Let (X', D') be the central fiber of $(\mathcal{X}' \to \mathcal{D}') \to \Delta$. The surface X' is the union of two components $X_1' \cup X_2'$, where X_i' is related to X_i by a diagram

$$X_i \leftarrow \widetilde{X}_i \to X_i',$$

given by Figure 5 where X_1 corresponds to T and X_2 corresponds to S.

Our next course of action differs substantially depending on the configuration of the directrices σ_1 and σ_2.

6.5.1. *Case (a):* σ_1 *and* σ_2 *do not intersect.* Let $p \in X_2$ be the intersection point of σ_1 with X_2. Since σ_1 and σ_2 are disjoint, σ_2 does not pass through p.

Step 2a (Contractions): We claim that $K_{\mathcal{X}'} + w\mathcal{D}'$ is nef.

To see this, it suffices to show that the restriction of $K_{\mathcal{X}'} + w\mathcal{D}'$ to each component of X' is nef. Consider the component X_1' of X'. Note that X_1' is obtained from $X_1 \cong \mathbb{F}_3$ by contracting the (-3) curve σ_1. Therefore, X_1' is of Picard rank 1, with $\mathrm{Pic}_{\mathbb{Q}}(X_1')$ generated by F, the image of the fiber of X_1. It is easy to calculate that

$$(K_{\mathcal{X}'} + w\mathcal{D}')\big|_{X_1'} = 6\epsilon F.$$

In particular, $K_{\mathcal{X}'} + w\mathcal{D}'$ is ample on X_1'. As a result, it suffices to show that $(K_{\mathcal{X}'} + w\mathcal{D}')|_{X_2'}$ is nef.

Note that $\sigma_2 \subset X_2'$ is a (-1)-curve lying in the smooth locus of X_2'. Let $X_2' \to X_2''$ be the contraction of σ_2. It is easy to see that both $K_{\mathcal{X}'}|_{X_2'}$ and D' are σ_2-trivial. Hence, they descend to Cartier divisors on X_2'', which we denote by K'' and D'', respectively. It now suffices to show that $K'' + wD''$ is nef on X_2''.

We now describe two extremal curves on X_2''. For the first, recall that $\widetilde{X}_2 \to X_2$ is the composite of three successive blow-ups, and $\widetilde{X}_2 \to X_2'$ contracts the exceptional divisors introduced in the first two of these three blow-ups. Let E be the image in X_2'' of the exceptional divisor of the third blow-up. For the second, note that there is a unique section τ of X_2 through p that is tangent to D_2 at p. Let L be the image in X_2'' of the proper transform of this section in $\widetilde{X}_2$.

Lemma 6.9.

(1) *The curves E and L generate the cone of curves $\overline{\mathrm{NE}}(X_2'')$.*

(2) *The divisor $K'' + wD''$ is nef. It is ample if τ is not triply tangent to D_2 at p.*

We say that τ is triply tangent to D_2 at p if the unique subscheme of D_2 of length 3 supported at p is contained in τ. In particular, if τ is a component of D_2, then it is triply tangent to D_2 at p.

Proof. It is easy to calculate the intersection table of E and L. If τ is not triply tangent to D_2 at p, we have the table

	E	L
E	$-\frac{1}{3}$	0
L	0	$-\frac{1}{3}$

,

and otherwise we have

	E	L
E	$-\frac{1}{3}$	1
L	1	-2

.

In either case, E and L represent effective classes of negative self-intersection, and therefore, they are extremal in $\overline{\mathrm{NE}}\left(X_2''\right)$. We also see that the classes of L and E are linearly independent. Since $\overline{\mathrm{NE}}\left(X_2''\right)$ is two-dimensional, it follows that L and E span it.

Let F be the image in X_2'' of the class of a fiber of X_2. Then we have $E \cdot F = 0$ and $L \cdot F = 1$. A straightforward computation shows that we have

$$K'' \equiv -2F + 2E, \text{ and}$$

$$D''\big|_{X_2''} \equiv 3F - 3E.$$

Therefore, we get

$$(K'' + wD'') \cdot E = \epsilon,, \text{ and}$$

$$(K'' + wD'') \cdot L = \begin{cases} 3\epsilon & \text{if } \tau \text{ is not triply tangent to } D_2 \text{ at } p, \\ 0 & \text{otherwise.} \end{cases}$$

The proof is now complete. □

With the proof of the lemma, we finish the proof that $K_{\mathcal{X}'} + w\mathcal{D}'$ is nef, and hence the two steps required for the proof of Theorem 6.1 in sub-case (a) of the $\mathbb{F}_3 - \mathbb{F}_1$ case. We recall that the stable limit $(\overline{X}, \overline{D})$ is obtained as the $(K_{\mathcal{X}'} + w\mathcal{D}')$-model of (X', D').

Remark 6.10. We record the geometry of $(\overline{X}, \overline{D})$ obtained above. Recall that $\overline{X}$ is obtained from X' by contracting the following curves: (1) the curve $\sigma_2 \subset X_2'$, and (2) the curve $L \subset X_2'$ if τ is triply tangent to D_2 at p.

If τ is triply tangent to D_2 at p, then we see that $\overline{X}$ is the union of $\overline{X}_1 = \mathbb{P}(3, 1, 1)$ and $\overline{X}_2 = \mathbb{P}(3, 1, 2)$, where the μ_3-singularity of $\overline{X}_1$ is glued to the A_2-singularity of $\overline{X}_2$ resulting in the (non-isolated) surface singularity p given by $xy = 0 \subset \frac{1}{3}(2, 1, 1)$. The A_1-singularity q of $\overline{X}_2$ lies away from the double curve. The divisor $\overline{D}$ lies away from p. If τ is not a component of D_2, then $\overline{D}$ lies away from q. If τ is a component of D_2, then $\overline{D}$ passes through q and has a node or a cusp there, depending on whether the residual curve $\overline{D_2 \setminus \tau}$ intersects τ transversely at two points or tangentially at one point.

If τ is not triply tangent to D_2 at p, then $\overline{X}$ is a smoothing of $\mathbb{P}(3, 1, 1) \cup \mathbb{P}(3, 1, 2)$ at the isolated A_1-singularity q. As in Remark 6.4, it is easy to check that the isomorphism type of $\overline{X}$ does not depend on the divisor D, and $\overline{X}$ is not a union of toric surfaces along toric subschemes.

6.5.2. *Case (b):* σ_1 *and* σ_2 *intersect.* In contrast with case (a), $K_{\mathcal{X}'} + w\mathcal{D}'$ is *not* nef in this case, and a further flip is necessary.

Step 1 (a further flip) in case (b): To perform the flip, we must understand the configuration of the curves σ_1, σ_2, and D_2. Let p be the point of intersection of σ_1 and σ_2. Since $\sigma_1 \subset D_1$, we must have $p \in D_2$. However, we also have $D_2 \cdot \sigma_2 = 0$, and therefore, we conclude that σ_2 must be a component of D_2. Let $D_2 = \sigma_2 \cup H$, where H is the residual curve. Then $H \subset X_2$ is a curve of class $2\sigma_2 + 3f$. Since D_2 is reduced, H does not contain σ_2 as a component, and since $H \cdot \sigma_2 = 1$, it must intersect σ_2 transversely

at a unique point q. Also, since D_2 intersects the fiber through p transversely, we have $q \neq p$. Let σ_2' be the proper transform of σ_2 in X_2'. Then σ_2' is a smooth rational curve of self-intersection (-4) in the smooth locus of X_2'. Let $\mathcal{X}' \dashrightarrow \mathcal{X}''$ be the type I flip along σ_2'. Let $\mathcal{D}''$ be the proper transform of $\mathcal{D}'$ in $\mathcal{X}''$.

Step 2 (contractions) in case(b): We are now ready for the contraction step. We claim that $K_{\mathcal{X}''} + w\mathcal{D}''$ is nef.

The proof of the nefness of $K_{\mathcal{X}''} + w\mathcal{D}''$ closely resembles the proof of nefness in case (a). As before, nefness on X_1'' is easy, using that X_1'' is of Picard rank 1. For X_2'', we have the diagram

$$\mathbb{F}_1 = X_2 \xleftarrow{a} \widetilde{X_2} \xrightarrow{b} X_2' \xleftarrow{a'} \widetilde{X_2'} \xrightarrow{b'} X_2'',$$

where the first transformation $X_2 \dashrightarrow X_2'$ is the result of a type II flip and the second transformation $X_2' \dashrightarrow X_2''$ is the result of a type I flip. That is, the map a consists of 3 successive blow-ups, b consists of 2 successive blow downs, a' consists of two successive blow-ups, and b' consists of 2 successive blow-downs. We may perform all the blow-ups first, followed by all the blow-downs, obtaining a sequence

$$X_2 \xleftarrow{\alpha} \Xi \xrightarrow{\beta} X_2''.$$

The exceptional locus of α consists of a chain of rational curves, whose dual graph is shown below.

Here, $\widetilde{\sigma_2}$ is the proper transform of σ_2. By contracting E_2, E_3, G_1 and G_2, we obtain X_2'; by contracting G_1, $\widetilde{\sigma_2}$, E_2 and E_3, we obtain X_2''.

Let $X_2'' \to X_2'''$ be the contraction of $\beta(E_1)$. Equivalently, let X_2''' be the surface obtained from Ξ by contracting the chain $G_1, \widetilde{\sigma_2}, E_1, E_2, E_3$. By contracting E_1 first, then E_2, then E_3, then the chain $G_1, \widetilde{\sigma_2}$, which are now both (-2) curves, we see that X_2''' has an A_2

singularity. It is easy to check that the divisors $K''_{X''}$ and D'' are both trivial on $\beta(E_1) \subset X''_2$, and hence both divisors are pull-backs of Cartier divisors from X'''_2, say K''' and D'''. It suffices to show that $K''' + wD'''$ is nef on X'''_2.

Denote by G the image in X'''_2 of G_2. Recall that $q \in X'_2$ is the intersection point of σ'_2 and H. Let F be the fiber of $X'_2 \to P_2$ through q, and let $\widetilde{F}$ be the proper transform of F in X'''_2. We have the following analogue of Lemma 6.9

Lemma 6.11.

   (1) The curves G and $\widetilde{F}$ generate the cone of curves $\overline{\mathrm{NE}}(X'''_2)$.

   (2) The divisor $(K''' + wD''')$ is nef. It is ample if F is not tangent to H at q.

We say that F is tangent to H at q if the unique subscheme of length 2 of H supported at q is contained in F. In particular, if H contains F as a component, then F is tangent to H at q.

Proof. The proof is analogous to the proof of Lemma 6.9. $\qquad\qquad\qquad\square$

With the proof of Lemma 6.11, the proof of nefness of $K_{X''} + w\mathcal{D}''$ is complete, and so are the two steps necessary for the proof of Theorem 6.1 in case (b). We recall that the stable limit $(\overline{X}, \overline{D})$ is obtained as the $(K_{X''} + w\mathcal{D}'')$-model of (X'', D'').

Remark 6.12. We observe that the pairs $(\overline{X}, \overline{D})$ obtained in case (b) are the same as the pairs $(\overline{X}, \overline{D})$ obtained in case (a).

More specifically, consider a pair (X', D') as in case (b). Let $(\mathbb{P}^2, C)$ be the plane cubic obtained from (X'_2, D'_2) by contracting σ_2, and let $L \subset \mathbb{P}^2$ be the image of the double curve in X'_2. Let X''_2 be the blow up of $\mathbb{P}^2$ at a general point of L, and let D''_2 be the proper transform of C in $\widetilde{X}_2$. Construct (X'', D'') by gluing (X'_1, D'_1) and (X''_2, D''_2) in the obvious way. Then (X'', D'') is pair as in case (a) that leads to the same stable limit $(\overline{X}, \overline{D})$ as in the pair (X', D').

Remark 6.13. Observe that if C_2 is smooth, then σ_1 and σ_2 must be disjoint as treated in § 6.5.1. In the resulting $(\overline{X}, \overline{D})$, the divisor $\overline{D}$ meets the double curve $\overline{X}_1 \cap \overline{X}_2$ at 2 distinct points q, r. The divisor $q + r$ is the hyperelliptic divisor of H_1.

To reconstruct (X, D) from $(\overline{X}, \overline{D})$ in this case, we must choose a point $t \in \overline{X}_1 \cap \overline{X}_2$ away from $\overline{D}$. The blow up of t on $\overline{X}_2$ yields X_2', and hence $X' = X_1' \cup X_2'$. We can then undo the transformations in the type 2 flip (§ 5.2) to obtain (X, D).

If we do the same procedure starting with t on $\overline{D}$, then then the corresponding (X, D) is a surface with intersecting directrices as in § 6.5.2.

6.6. **Summary of stable replacements.** Thanks to the proof of Theorem 6.1 in Section 6, we obtain an explicit list of stable log quadrics (S, D), namely the points of $\mathfrak{X}$.

We first look at the surfaces S. Table 1 lists the possible surfaces S along with its non-normal-crossing singularities. If S is reducible, then Table 1 also describes the double curve on each component. In the table, the divisor H on a weighted projective space refers to the zero locus of a section of the primitive ample line bundle, and the divisor F on (coarse space of) a projective bundle denotes the (coarse space of) a fiber. The last column directs the reader to the relevant section in Section 6 where the stable reduction is obtained.

Remark 6.14. The surface S described as the coarse space of $\mathbb{P}(\mathcal{O}(4/3, 5/3) \oplus \mathcal{O}(5/3, 4/3))$ has two alternate descriptions (see Remark 6.7). First, it is obtained by gluing $\mathrm{Bl}_u \mathbb{P}(3, 1, 1)$ and $\mathrm{Bl}_v \mathbb{P}(3, 1, 1)$ along a $\mathbb{P}^1$, where u and v are curvilinear subschemes of length 3. Second, it is a degenerate (non-normal) toric surface represented by the subdivided rectangle in Figure 9.

We now look at the divisors D. By Remark 4.3, the curve D is reduced and only admits A_m singularities for $m \leq 4$. We also observe that D is a Cartier divisor. In particular, the log quadrics (S, D) satisfy the index condition. To see that D is Cartier, it suffices

to examine it locally at the singular points of S. We observe that whenever D passes through an isolated singularity of S, it is an A_1 singularity; D is either nodal or cuspidal at the singularity, and is cut out by one equation. Whenever D passes through a non-isolated singularity of S, it does so at the transverse union of two smooth surfaces; the local picture of (S, D) is

$$(\operatorname{spec} \mathbb{K}[x, y, t]/(xy), t).$$

Thus, D is Cartier. Furthermore, we can check directly that (S, D) satisfies the definition of a stable log surface for all positive $\epsilon < 1/30$.

We collect the observations above in the following theorem.

Theorem 6.15. *Let (S, D) be a stable log quadric.*

(1) The isomorphism class of S is one of the 8 listed in Table 1.

(2) The divisor D is Cartier. In particular, (S, D) satisfies the index condition (Definition 3.4).

S	Singularities of S	Double curve	Reference
$\mathbb{P}^1 \times \mathbb{P}^1$	$-$	$-$	$-$
$\mathbb{P}(1, 1, 2)$	$p : A_1$	$-$	§ 6.1
$\mathbb{Q}$-Gorenstein smoothing of the A_1 singularity of $\mathbb{P}(9, 1, 2)$	$p : \frac{1}{9}(1, 2)$	$-$	§ 6.2 Case (1)
$\mathbb{P}(9, 1, 2)$	$p : \frac{1}{9}(1, 2), q : A_1$	$-$	§ 6.2 Case (2), (3)
Coarse space of $\mathbb{P}(\mathcal{O}(4/3, 5/3) \oplus \mathcal{O}(5/3, 4/3))$	$p : (xy = 0) \subset \frac{1}{3}(1, 2, 1), q : (xy = 0) \subset \frac{1}{3}(2, 1, 1)$	F, F	§ 6.3
$\mathbb{P}^2 \cup \mathbb{P}^2$	$(xy = 0) \subset \mathbb{A}^3$	H, H	§ 6.4
$\mathbb{Q}$-Gorenstein smoothing of the A_1 singularity of $\mathbb{P}(3, 1, 2) \cup \mathbb{P}(3, 1, 1)$	$p : (xy = 0) \subset \frac{1}{3}(2, 1, 1)$	deformation of $2H$, deformation of H	§ 6.5 (non triply tangent case)
$\mathbb{P}(3, 1, 2) \cup \mathbb{P}(3, 1, 1)$	$p : (xy = 0) \subset \frac{1}{3}(2, 1, 1), q : A_1$ on $\mathbb{P}(3, 1, 2)$	$2H, H$	§ 6.5 (triply tangent case)

TABLE 1. Surfaces S that appear in stable log quadrics (S, D)

(3) (S, D) *satisfies Definition 3.1 for all positive $\epsilon < 1/30$.*

As a corollary, we obtain the following.

Corollary 6.16. *The stack $\mathfrak{X}$ is of finite type and proper over $\mathbb{K}$.*

Proof. From Theorem 6.15 (3), we get that $\mathfrak{X}$ is a locally closed substack of the finite type stack $\mathfrak{F}_{\epsilon,8}$ for a positive $\epsilon < 1/30$ (see Proposition 3.8). The valuative criterion for properness follows from the valuative criterion for properness for $\overline{\mathcal{H}}_{4,6}(1/6 + \epsilon)$ [12, Corollary 6.6] and stabilization (Theorem 6.1). $\qquad\square$

We take a closer look at the pairs (S, D) where D is smooth. We see that these arise from a triple cover $f : C \to \mathbb{P}^1$ where C is smooth non-hyperelliptic curve of genus 4, or from $g : C \cup_p \mathbb{P}^1 \to \mathbb{P}^1$ where C is a smooth hyperelliptic curve of genus 4.

Corollary 6.17. *For all (S, D) such that D is smooth, we have the following classification by Table 2:*

S	D	Embedding $D \hookrightarrow S$
$\mathbb{P}^1 \times \mathbb{P}^1$	Non-hyperelliptic, Maroni general	Induced by the canonical embedding
$\mathbb{P}(1,1,2)$	Non-hyperelliptic, Maroni special	Induced by the canonical embedding
$\mathbb{Q}$-Gorenstein smoothing of the A_1 singularity of $\mathbb{P}(9,1,2)$	Hyperelliptic	Determined by a hyperelliptic divisor $p + q$ with $p \neq q$
$\mathbb{P}(9,1,2)$	Hyperelliptic	Determined by a hyperelliptic divisor $2p$.

TABLE 2. Surfaces S that appear in stable log quadrics (S, D) with D smooth

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7. DEFORMATION THEORY

In this section, we study the $\mathbb{Q}$-Gorenstein deformations of pairs parametrized by $\mathfrak{X}$. Our treatment closely follows [20, § 3]. Since many of the results carry over from [20, § 3] our treatment will be brief.

7.1. **The $\mathbb{Q}$-Gorenstein cotangent complex.** Let A be an affine scheme, and $S \to A$ a $\mathbb{Q}$-Gorenstein family of surfaces. Denote by $p \colon \mathcal{S} \to S$ the canonical covering stack of S. By the definition of a $\mathbb{Q}$-Gorenstein family, $\mathcal{S} \to A$ is flat. Let $L_{\mathcal{S}/A}$ be the cotangent complex of $\mathcal{S} \to A$ [27].

Definition 7.1 ($\mathbb{Q}$-Gorenstein deformation functors)**.** Let M be a quasi-coherent $\mathcal{O}(A)$-module. Define the $\mathcal{O}(A)$-module $T^i_{\mathbb{Q}\mathrm{Gor}}(S/A, M)$ and the $\mathcal{O}_{\mathcal{S}}$-module $\mathcal{T}^i_{\mathbb{Q}\mathrm{Gor}}(S/A, M)$ by

$$T^i_{\mathbb{Q}\mathrm{Gor}}(S/A, M) = \mathrm{Ext}^i(L_{\mathcal{S}/A}, \mathcal{O}_{\mathcal{S}} \otimes_A M),$$

$$\mathcal{T}^i_{\mathbb{Q}\mathrm{Gor}}(S/A, M) = p_* \mathcal{E}xt^i(L_{\mathcal{S}/A}, \mathcal{O}_{\mathcal{S}} \otimes_A M).$$

Recall that we also have the usual deformation functors $T^i(S/A, M)$ and $\mathcal{T}^i(S/A, M)$ defined using the cotangent complex of $S \to A$. The usual functors, in general, differ from the $\mathbb{Q}$-Gorenstein ones (except for $i = 0$, see Theorem 7.2).

The $\mathbb{Q}$-Gorenstein deformation functors play the expected role in classifying $\mathbb{Q}$-Gorenstein deformations and obstructions. To make this precise, let $A \to A'$ be an infinitesimal extension of A. A $\mathbb{Q}$-*Gorenstein deformation* of $S \to A$ over A' is a flat morphism $S' \to A'$ along with an isomorphism $S' \times_{A'} A \cong S$.

Let $A \to A'$ be a square zero extension of A by a quasi-coherent $\mathcal{O}(A)$-module M. Recall that this means we have a surjection $\mathcal{O}(A') \to \mathcal{O}(A)$ with kernel M and $M^2 = 0$.

Theorem 7.2. *Let $S \to A$ be a $\mathbb{Q}$-Gorenstein family of surfaces and let $A \to A'$ be a square zero extension by an A-module M.*

(1) *There is a canonical element $o(S/A, A') \in T^2_{\mathbb{Q}\mathrm{Gor}}(S/A, M)$ which vanishes if and only if there exists of $\mathbb{Q}$-Gorenstein deformation of S/A over A'.*

(2) *If $o(S/A, A') = 0$, then the set of isomorphism classes of $\mathbb{Q}$-Gorenstein deformations of S/A over A' is an affine space under $T^1_{\mathbb{Q}\mathrm{Gor}}(S/A, M)$.*

(3) *If S'/A' is a $\mathbb{Q}$-Gorenstein deformation of S/A, then the group of automorphisms of S' over A' that restrict to the identity on S is isomorphic to $T^0_{\mathbb{Q}\mathrm{Gor}}(S/A, M)$. Furthermore, we have an isomorphism*

(7.1) $$\mathcal{T}^0_{\mathbb{Q}\mathrm{Gor}}(S/A, M) \cong \mathcal{T}^0(S/A, M).$$

Proof. The isomorphism (7.1) is from [20, Lemma 3.8]. The rest of the assertions are from [20, Theorem 3.9]. The main point in the proof is an equivalence between $\mathbb{Q}$-Gorenstein deformations of S and deformations of $\mathcal{S}$. Having established this equivalence, the theorem follows from the properties of the cotangent complex [27, Theorem 1.7]. □

7.2. **Deformations of pairs.** Having discussed deformations of surfaces, we turn to deformations of pairs. The upshot of this discussion is Proposition 7.5, which says that the deformations of pairs are no more challenging than the deformations of the ambient surfaces.

Let (S, D) be a stable log quadric, that is, a $\mathbb{K}$-point of $\mathfrak{X}$. The $\mathbb{Q}$-Gorenstein cotangent complex of a surface S is determined by the canonical covering stack $p\colon \mathcal{S} \to S$. We collect the properties of $\mathcal{S}$ that we require for further analysis. Set $D_{\mathcal{S}} = D \times_S \mathcal{S}$.

Lemma 7.3. *The stack $\mathcal{S}$ has lci singularities.*

Proof. Recall that $\mathcal{S} \to S$ is an isomorphism over the Gorenstein locus of in S. From Theorem 6.15, we see that the only non-Gorenstein singularities on S are $\frac{1}{3}(1, 1)$, $\frac{1}{9}(1, 2)$, and $(xy = 0) \subset \frac{1}{3}(2, 1, 1)$, and furthermore, all other singularities of S are lci. The

canonical covering stacks of the three non-Gorenstein singularities are

$$\left[\mathbb{A}^2/\mu_3\right] \to \frac{1}{3}(1,1),$$

$$\left[\operatorname{spec}\mathbb{K}[x,y,z]/(xy-z^3)/\mu_3\right] \to \frac{1}{9}(1,1), \text{ and}$$

$$\left[\operatorname{spec}\mathbb{K}[x,y,z]/(xy)/\mu_3\right] \to (xy=0) \subset \frac{1}{3}(2,1,1).$$

All three stacks on the left have lci (in fact, hypersurface) singularities. The first assertion follows. □

Lemma 7.4. *Let (S,D) be a stable log quadric. Then $H^1(\mathcal{O}_S(D)) = 0$.*

Proof. The assertion is analogous to [20, Lemma 3.14]. The same proof goes through as long as we check that $-(K_S - D)$ is ample and the normalization S^ν is log terminal. From Theorem 6.15, we know that $D \cong -3/2K_S$ and $K_S + (2/3 + \epsilon)D$ is ample. It follows that both $-K_S$ and D are ample, and hence so is $-(K_S - D)$. From looking at the singularities of S in Theorem 6.15, we see that S^ν is log terminal. □

Proposition 7.5. *Let A be an affine scheme and (S,D) an object of $\mathfrak{X}(A)$. Let $A \to A'$ be an infinitesimal extension and $S' \to A'$ a $\mathbb{Q}$-Gorenstein deformation of S/A. Then there exists a $\mathbb{Q}$-Gorenstein deformation (S', D') over A' of (S,D). That is, there exists an object of $\mathfrak{X}(A')$ restricting to (S,D) over A.*

Proof. The assertion is analogous to [20, Theorem 3.12]. The proof depends on two lemmas: [20, Lemma 3.13] and [20, Lemma 3.14]. The analogue of the first is Theorem 6.15(2) and of the second is Lemma 7.4. □

We now have all the tools to show that the $\mathbb{Q}$-Gorenstein deformations of stable log quadrics are unobstructed.

Theorem 7.6. *$\mathfrak{X}$ over $\mathbb{K}$ is a smooth stack.*

Proof. We use the infinitesimal lifting criterion for smoothness. Let (S, D) be a stable log quadric. Let A be the spectrum of an Artin local $\mathbb{K}$-algebra, $(\mathscr{S}, \mathscr{D})$ be a $\mathbb{Q}$-Gorenstein deformation of (S, D) over A, and $A \to A'$ an infinitesimal extension. We must show that $(\mathscr{S}, \mathscr{D})$ extends to a deformation $(\mathscr{S}', \mathscr{D}')$ over A'.

By induction on the length, it suffices to prove the statement when the kernel of $\mathcal{O}(A') \to \mathcal{O}(A)$ is $\mathbb{K}$. By Proposition 7.5, it suffices to show the existence of $\mathscr{S}'$. By Theorem 7.2, it suffices to show that $T^2_{\mathbb{Q}\mathrm{Gor}}(\mathscr{S}/A, k) = 0$. Note that we have $T^2_{\mathbb{Q}\mathrm{Gor}}(\mathscr{S}/A, k) = T^2_{\mathbb{Q}\mathrm{Gor}}(S/k, k)$. Henceforth, we abbreviate $T^i_{\mathbb{Q}\mathrm{Gor}}(S/k, k)$ by $T^i_{\mathbb{Q}\mathrm{Gor}}(S)$ and use similar abbreviations for T^i and $\mathcal{T}^i$.

To show that $T^2_{\mathbb{Q}\mathrm{Gor}} = 0$, it suffices to show by the Leray spectral sequence that $H^0\left(\mathcal{T}^2_{\mathbb{Q}\mathrm{Gor}}\right)$, $H^1\left(\mathcal{T}^1_{\mathbb{Q}\mathrm{Gor}}\right)$, and $H^2\left(\mathcal{T}^0_{\mathbb{Q}\mathrm{Gor}}\right)$ are all 0. We do this one by one.

Let $p\colon \mathcal{S} \to S$ be the canonical covering stack. By Lemma 7.3, we know that $\mathcal{S}$ is lci. Therefore, $\mathcal{T}^2(\mathcal{S}) = 0$, and hence $\mathcal{T}^2_{\mathbb{Q}\mathrm{Gor}}(S) = 0$.

The sheaf $\mathcal{T}^1_{\mathbb{Q}\mathrm{Gor}}(S)$ is supported on the singular locus of S. If the singular locus has dimension less than one, then $H^1(\mathcal{T}^1_{\mathbb{Q}\mathrm{Gor}}(S)) = 0$. From Theorem 6.15, we see that the only cases where the singular locus of S has dimension ≥ 1 have $S = S_1 \cup_B S_2$, where S_1 and S_2 are irreducible and meet along a curve $B \cong \mathbb{P}^1$. More precisely, the local structure of S is either $(xy = 0) \subset \mathbb{A}^3$ or its quotient by a μ_r where $\zeta \in \mu_r$ acts by

$$\zeta \cdot (x, y, z) \mapsto (\zeta x, \zeta^{-1} y, \zeta^a z),$$

with $\gcd(a, r) = 1$. Denote by B_i the restriction of B to S_i for $i = 1, 2$. By [22, Proposition 3.6], we get that in this case

$$\mathcal{T}^1_{\mathbb{Q}\mathrm{Gor}} = \mathcal{O}_{S_1}(B_1)\big|_B \otimes \mathcal{O}_{S_2}(B_2)\big|_B.$$

Thus, $\mathcal{T}^1_{\mathbb{Q}\mathrm{Gor}}$ is a line bundle on $B \cong \mathbb{P}^1$ of degree $B_1^2 + B_2^2$. In the surfaces listed in Theorem 6.15, we see that $B_1^2 + B_2^2$ is either 0, 1, or 2. We conclude that $H^1(\mathcal{T}^1_{\mathbb{Q}\mathrm{Gor}}) = 0$.

By Theorem 7.2 equation (7.1), the sheaf $\mathcal{T}^0_{\mathbb{Q}\mathrm{Gor}}(S)$ is isomorphic to $\mathcal{T}^0(S)$. First, assume that S is reducible with the notation as above. Then [20, Lem 9.4] applies to S as its proof is valid whenever S is slc, S_i only has quotient singularities, S is not normal crossing along at most two points of B, the divisor $K_{S_i} + B_i$ anti-ample, and $h^1(\mathcal{O}_{\tilde{S}_i}) = 0$ where $\tilde{S}_i \to S_i$ are the minimal resolutions. Of these, the first three conditions follow from Theorem 6.15. The anti-ampleness of $K_{S_i} + B_i$ can be seen by noting that they are restrictions of the anti-ample $\mathbb{Q}$-divisor K_S to S_i. Finally, since each S_i is rational by Theorem 6.15, we have $h^1(\mathcal{O}_{\tilde{S}_i}) = 0$. This proves that $H^2(\mathcal{T}^0_S) = 0$ whenever S is reducible.

When S is irreducible, consider the minimal resolution $c \colon \tilde{S} \to S$. Since S only has quotient singularities by Theorem 6.15, the surface $\tilde{S}$ is rational as well by [30, Prop 5.15]. Therefore, we have $c_* \mathcal{O}_{\tilde{S}} = \mathcal{O}_S$ and $R^i c_* \mathcal{O}_{\tilde{S}} = 0$ for any $i > 0$. Furthermore, since $\tilde{S}$ is rational, we have $q(S) := h^1(\mathcal{O}_S) = h^1(c_* \mathcal{O}_{\tilde{S}}) = 0$ and $p_g(S) := h^2(\mathcal{O}_S) = h^2(c_* \mathcal{O}_{\tilde{S}}) = 0$. Since $-K_S$ is ample and effective as well, [32, Prop III.5.3] implies that $H^2(\mathcal{T}^0_S) = 0$.

The proof of Theorem 7.6 is thus complete. $\qquad\square$

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In this section, we take a closer look at the geometry of $\mathcal{X}$, and compare it with related moduli spaces.

8.1. Comparison of $\mathcal{X}$ with the spaces of weighted admissible covers $\overline{\mathcal{H}}_4^3(1/6 + \epsilon)$.

Recall that $\overline{\mathcal{H}}_4^3(1/6 + \epsilon)$ is the moduli space of weighted admissible covers where up to 5 branch points are allowed to coincide. Let $U \subset \overline{\mathcal{H}}_4^3(1/6 + \epsilon)$ be the open substack parametrizing $\phi \colon C \to P$ where $P \cong \mathbb{P}^1$, the curve C is smooth, and the Tschirnhausen bundle E_ϕ of ϕ is $\mathcal{O}(3) \oplus \mathcal{O}(3)$. We have a morphism $\Phi \colon U \to \mathcal{X}$ given by the transformation

$$(\phi \colon C \to P) \mapsto (\mathbb{P}E_\phi, C).$$

Theorem 8.1. *The map Φ extends to a morphism of stacks $\Phi \colon \overline{\mathcal{H}}_4^3(1/6 + \epsilon) \to \mathcal{X}$.*

Since $\overline{\mathcal{H}}_4^3(1/6 + \epsilon)$ is proper and $\mathcal{X}$ is separated, the map Φ is also proper.

For the proof, we need extension lemmas for morphisms of stacks, extending some well-known results for schemes. Let $\mathcal{X}$ and $\mathcal{Y}$ be separated Deligne–Mumford stacks of finite type over a field $\mathbb{K}$. Let $\mathcal{U} \subset \mathcal{X}$ be a dense open substack.

Lemma 8.2. *Assume that $\mathcal{X}$ is normal. If $\Phi_1, \Phi_2 \colon \mathcal{X} \to \mathcal{Y}$ are two morphisms whose restrictions to $\mathcal{U}$ are equal (2-isomorphic), then Φ_1 and Φ_2 are equal (2-isomorphic).*

Proof. We have the following diagram where the square is a pull-back

$$
\begin{array}{ccc}
\mathcal{Z} & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \Delta \\
\mathcal{U} \to \mathcal{X} & \xrightarrow{(\Phi_1, \Phi_2)} & \mathcal{Y} \times \mathcal{Y}
\end{array}
$$

Since $\mathcal{Y}$ is a separated Deligne–Mumford stack, the diagonal map Δ is representable, proper, and unramified. Therefore, so is the pullback $\mathcal{Z} \to \mathcal{X}$. Since Φ_1 and Φ_2 agree on $\mathcal{U}$, the inclusion $\mathcal{U} \to \mathcal{X}$ lifts to $\mathcal{U} \to \mathcal{Z}$. Since $\mathcal{Z} \to \mathcal{X}$ is unramified and $\mathcal{U} \to \mathcal{X}$ is an

open immersion, so is the lift $\mathcal{U} \to \mathcal{Z}$. Let $\overline{\mathcal{U}} \subset \mathcal{Z}$ be the closure of $\mathcal{U}$ and $\overline{\mathcal{U}}^\nu \to \overline{\mathcal{U}}$ its normalization. Since $\mathcal{X}$ is normal, Zariski's main theorem implies that $\overline{\mathcal{U}}^\nu \to \mathcal{X}$ is an isomorphism. Hence the map $\mathcal{Z} \to \mathcal{X}$ admits a section $\mathcal{X} \to \mathcal{Z}$. In other words, the map $(\Phi_1, \Phi_2)\colon \mathcal{X} \to \mathcal{Y} \times \mathcal{Y}$ factors through the diagonal $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$. $\qquad\square$

Example 8.3. In Lemma 8.2, we can drop the normality assumption on $\mathcal{X}$ if $\mathcal{Y}$ is an algebraic space, but not otherwise. An example of distinct maps that agree on a dense open substack can be constructed using twisted curves (see [1, Proposition 7.1.1]). Let $\mathcal{X}$ be the stack

$$\mathcal{X} = [\operatorname{spec} \mathbb{C}[x, y]/xy/\mu_n],$$

where $\zeta \in \mu_n$ acts by $(x, y) \mapsto (\zeta x, \zeta^{-1} y)$. Every $\zeta \in \mu_n$ defines an automorphism of $t_\zeta\colon \mathcal{X} \to \mathcal{X}$ given by $(x, y) \mapsto (x, \zeta y)$. The map t_ζ is the identity map on the complement of the node of $\mathcal{X}$, but not the identity map on $\mathcal{X}$ if $\zeta \neq 1$.

The map $\Phi\colon \mathcal{U} \to \mathcal{Y}$ induces a map $|\Phi|\colon |\mathcal{U}| \to |\mathcal{Y}|$ on the set of points. Let $\phi\colon |\mathcal{X}| \to |\mathcal{Y}|$ be an extension of $|\Phi|$. We say that ϕ is *continuous in one-parameter families* if for every DVR Δ and a map $i\colon \Delta \to \mathcal{X}$ that sends the generic point η of Δ to $\mathcal{U}$, the map $\Phi \circ i\colon \eta \to \mathcal{Y}$ extends to a map $\Delta \to \mathcal{Y}$ and agrees with the map ϕ on the special point.

Lemma 8.4. *Suppose $\mathcal{X}$ is smooth, and $\Phi\colon \mathcal{U} \to \mathcal{Y}$ is a morphism. Let $\phi\colon |\mathcal{X}| \to |\mathcal{Y}|$ be an extension of $|\Phi|\colon |\mathcal{U}| \to |\mathcal{Y}|$. If ϕ is continuous in one-parameter families, then it is induced by a morphism $\Phi\colon \mathcal{X} \to \mathcal{Y}$ that extends $\Phi\colon \mathcal{U} \to \mathcal{Y}$.*

By Lemma 8.2, the extension is unique.

Proof. Consider the map $(\operatorname{id}, \Phi)\colon \mathcal{U} \to \mathcal{X} \times \mathcal{Y}$. Let $\mathcal{Z} \subset \mathcal{X} \times \mathcal{Y}$ be the scheme theoretic image of $\mathcal{U}$ (see [40, Tag 0CMH]), and let let $\mathcal{Z}^\nu \to \mathcal{Z}$ be the normalization. By construction, the map $\mathcal{Z} \to \mathcal{X}$ is an isomorphism over $\mathcal{U}$ [40, Tag 0CPW]. Since $\mathcal{U}$ is smooth (and hence normal), the map $\mathcal{Z}^\nu \to \mathcal{X}$ is also an isomorphism over $\mathcal{U}$. Our aim is to show that $\mathcal{Z}^\nu \to \mathcal{X}$ is in fact an isomorphism.

Let $Z^{\nu} \to X$ be the morphism on coarse spaces induced by $\mathcal{Z}^{\nu} \to \mathcal{X}$. Let $(x, y) \in |\mathcal{X}| \times |\mathcal{Y}|$ be a point from Z^{ν}. Since the image of $\mathcal{U}$ is dense in $\mathcal{Z}^{\nu}$, there exists a DVR Δ with a map $\Delta \to \mathcal{Z}^{\nu}$ whose generic point maps into the image of $\mathcal{U}$ and whose special point maps to (x, y). The continuity of ϕ in one-parameter families implies that $y = \phi(x)$. As a result, $Z^{\nu} \to X$ is a bijection on points. As Z^{ν} and X are normal spaces, $Z^{\nu} \to X$ must be an isomorphism.

By hypothesis, for every DVR Δ, a map $\Delta \to \mathcal{X}$ that sends the generic point to $\mathcal{U}$ lifts to a map $\Delta \to \mathcal{Y}$, and hence to a map $\Delta \to \mathcal{Z}^{\nu}$. This implies that $\mathcal{Z}^{\nu} \to \mathcal{X}$ is unramified in codimension 1. It follows by the same arguments as in [17, Corollary 6] that $\mathcal{Z}^{\nu} \to \mathcal{X}$ is an isomorphism. Since [17, Corollary 6] is stated with slightly stronger hypotheses, we recall the proof. Let V be a scheme and $V \to \mathcal{X}$ an étale morphism. Set $\mathcal{W} = \mathcal{Z}^{\nu} \times_{\mathcal{X}} V$ and $U = \mathcal{U} \times_{\mathcal{X}} V$. Let $\mathcal{W} \to W$ be the coarse space. The map $W \to V$ is an isomorphism over the dense open subset $U \subset V$, and is a quasi-finite map between normal spaces. By Zariski's main theorem, it is an isomorphism. Furthermore, as $\mathcal{W} \to V$ is unramified in codimension 1, so is $\mathcal{W} \to W$. Since $\mathcal{W}$ is normal and W is smooth, purity of the branch locus [40, Tag 0BMB] implies that $\mathcal{W} \to W$ is étale. As $\mathcal{W} \to V$ is an isomorphism over U, we see that $\mathcal{W}$ contains a copy of U as a dense open substack. In particular, $\mathcal{W}$ has trivial generic stabilizers. By [17, Lemma 4], we conclude that $\mathcal{W} \to W$ is an isomorphism. Since both $\mathcal{W} \to W$ and $W \to V$ are isomorphisms, their composite $\mathcal{W} \to V$ is an isomorphism. We have proved that $\mathcal{Z}^{\nu} \to \mathcal{X}$ is an isomorphism étale locally on $\mathcal{X}$. We conclude that $\mathcal{Z}^{\nu} \to \mathcal{X}$ is an isomorphism.

The composite of the inverse of $\mathcal{Z}^{\nu} \to \mathcal{X}$, the map $\mathcal{Z}^{\nu} \to \mathcal{X} \times \mathcal{Y}$, and the projection onto $\mathcal{Y}$ gives the required extension $\Phi \colon \mathcal{X} \to \mathcal{Y}$. $\qquad\square$

Proof of Theorem 8.1. Define a map $\phi \colon |\overline{\mathcal{H}}_4^3(1/6 + \epsilon)| \to |\mathcal{X}|$, consistent with the map induced by Φ on U as follows. Let k be an algebraically closed field and $a \colon \operatorname{spec} k \to \overline{\mathcal{H}}_4^3(1/6 + \epsilon)$ a map. Let Δ be a DVR with residue field k and $\alpha \colon \Delta \to \overline{\mathcal{H}}_4^3(1/6 + \epsilon)$ a map that restricts to a at the special point and maps the generic point η to U. By

Theorem 6.1, there exists an extension $\beta\colon \Delta \to \mathfrak{X}$ of $\Phi \circ \alpha|_\eta$. Let $b\colon \operatorname{spec} k \to \mathfrak{X}$ be the central point of the extension. Set $\phi(a) = b$. Theorem 6.1 guarantees that b depends only on a, and not on α; so the map ϕ is well-defined. Theorem 6.1 also guarantees that ϕ is continuous in one-parameter families. Since $\overline{\mathcal{H}}_4^3(1/6 + \epsilon)$ is smooth, Lemma 8.4 applies and yields the desired extension. $\qquad\qquad\square$

8.2. **The boundary locus of $\mathfrak{X}$.** Let $\mathfrak{U} \subset \mathfrak{X}$ be the open subset that parametrizes (S, D) where $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $D \subset S$ is a smooth curve of degree $(3,3)$. The *boundary* of $\mathfrak{X}$ refers to the complement $\mathfrak{X} \setminus \mathfrak{U}$. Let $\mathfrak{T} \subset \overline{\mathcal{H}}_4^3(1/6 + \epsilon)$ be the open subset that parametrizes $f\colon C \to P$ where $P \cong \mathbb{P}^1$, the curve C is smooth, and the Tschirnhausen bundle of f is $\mathcal{O}(3) \oplus \mathcal{O}(3)$. We see that $\mathfrak{T} = \Phi^{-1}(\mathfrak{U})$. To understand the boundary of $\mathfrak{X}$, we are led to understanding $\overline{\mathcal{H}}_4^3(1/6 + \epsilon) \setminus \mathfrak{T}$.

We define some closed subsets of $\overline{\mathcal{H}}_4^3(1/6 + \epsilon) \setminus \mathfrak{T}$. Before we do so, let us extend the notion of the Maroni invariant of a triple cover of $\mathbb{P}^1$ to a triple cover of $P = \mathbb{P}^1(\sqrt[a]{p})$. Recall that vector bundles on P are direct sums of line bundles, and the line bundles are given by $\mathcal{O}(n)$ for $n \in \frac{1}{a}\mathbb{Z}$, where the generator $\mathcal{O}(-1/a)$ is the ideal sheaf of the unique stacky point p [33].

Definition 8.5 (Maroni invariant). Let $P = \mathbb{P}^1(\sqrt[a]{p})$, and let $f\colon C \to P$ be a representable, finite, flat morphism of degree 3. Suppose $f_*\mathcal{O}_C / \mathcal{O}_P \cong \mathcal{O}(-m) \oplus \mathcal{O}(-n)$ for some $m, n \in \frac{1}{a}\mathbb{Z}$. Then the *Maroni invariant* of f, denoted by $M(f)$, is the difference $|m - n|$.

We now define various boundary loci of $\overline{\mathcal{H}}_4^3(1/6 + \epsilon)$ based on the Maroni invariant and the singularities.

Definition 8.6 (Tschirnhausen loci). Let a, b, c be positive rational numbers. Define the following closed subsets of $\overline{\mathcal{H}}_4^3(1/6 + \epsilon)$.

$$\mathfrak{Y}_0 := \overline{\{[f : C \to P] \mid P \cong \mathbb{P}^1,\ M(f) = 0,\ C' \text{ is singular}\}}$$

$$\mathfrak{Y}_a := \overline{\{[f : C \to P] \mid P \cong \mathbb{P}^1,\ M(f) = a\}}$$

$$\mathfrak{Y}_{b,c} := \overline{\{[f : C \to P] \mid P \text{ is rational chain of length 2},\ M(f) \text{ on each component is } b, c\}}$$

Define $\mathfrak{Z}_a$, $\mathfrak{Z}_{b,c}$ to be the image of $\mathfrak{Y}_a$, $\mathfrak{Y}_{b,c}$ under Φ, respectively.

Since Φ is a proper map, $\mathfrak{Z}_a$ and $\mathfrak{Z}_{b,c}$ are closed subsets of $\mathfrak{X}$. We have seen in Section 6 that the corresponding cases there describe general members of $\mathfrak{Z}_a$ and $\mathfrak{Z}_{b,c}$ for suitable a, b, c. By construction, the various $\mathfrak{Z}_a$ and $\mathfrak{Z}_{b,c}$ cover the boundary of $\mathfrak{X}$ as a, b, c vary.

Let us identify a, b, c that lead to non-empty loci in $\overline{\mathcal{H}}_4^3(1/6 + \epsilon)$. Let us start with $\mathfrak{Y}_a$, keeping in mind that a must be even. Taking $a = 2$ yields the classical Maroni divisor $\mathfrak{Y}_2$. Taking $a = 4$ yields the hyperelliptic divisor $\mathfrak{Y}_4$. A generic point of $\mathfrak{Y}_4$ corresponds to $f : C \cup \mathbb{P}^1 \to \mathbb{P}^1$, where C is a smooth hyperelliptic curve of genus 4 attached to $\mathbb{P}^1$ nodally at one point. For $a > 4$, we have $\mathfrak{Y}_a = \varnothing$.

Let us now consider $\mathfrak{Y}_{b,c}$. First, observe that the node on the rational chain P of length 2 has automorphism group of order 1 or 3. In the case of trivial automorphism group, the non-empty cases are $\mathfrak{Y}_{1,1}$, $\mathfrak{Y}_{1,3}$, and $\mathfrak{Y}_{3,3}$. A generic point of $\mathfrak{Y}_{1,1}$ corresponds to $f : C_1 \cup C_2 \to P_1 \cup P_2$ where each C_i is a smooth curve of genus 1. A generic point of $\mathfrak{Y}_{1,3}$ corresponds to $f : C_1 \cup C_2 \to P_1 \cup P_2$ where C_1 is a smooth curve of genus 1 and C_2 is the disjoint union of a smooth curve of genus 2 and $\mathbb{P}^1$. A generic point of $\mathfrak{Y}_{3,3}$ corresponds to $f : C_1 \cup C_2 \to P_1 \cup P_2$ where each C_i is a disjoint union of a smooth curve of genus 2 and $\mathbb{P}^1$; they are attached so that the union $C_1 \cup C_2$ is connected. In the case of an automorphism group of order 3, the only non-empty case is $\mathfrak{Y}_{1/3,1/3}$. A generic point of $\mathfrak{Y}_{1/3,1/3}$ corresponds to $f : C_1 \cup C_2 \to P_1 \cup P_2$ where C_i is smooth of genus 2, and

on the level of coarse spaces, $C_i \to P_i$ is a triple cover totally ramified over the node point on P_i.

From the discussion above, we see that the non-empty $\mathfrak{Y}_a$ and $\mathfrak{Y}_{b,c}$, namely $\mathfrak{Y}_0$, $\mathfrak{Y}_2$, $\mathfrak{Y}_4$, $\mathfrak{Y}_{1,1}$, $\mathfrak{Y}_{1,3}$, $\mathfrak{Y}_{3,3}$, and $\mathfrak{Y}_{1/3,1/3}$, are all irreducible of codimension 1 in $\overline{\mathcal{H}}_4^3(1/6 + \epsilon)$. Set

$$I = \{0, 2, 4, (1,1), (1,3), (3,3), (1/3, 1/3)\}.$$

This is the set of possible subscripts of the $\mathfrak{Y}$'s.

Proposition 8.7. *For all $i \in I$ except $i = (1,1)$ and $i = (1,3)$, the loci $\mathfrak{Z}_i$ are of codimension 1 in $\mathfrak{X}$. The locus $\mathfrak{Z}_{1,1}$ is of codimension 3, and $\mathfrak{Z}_{1,3}$ of codimension 2.*

Proof. For all $i \in I$, the $\mathfrak{Y}_i$ are irreducible, and hence so are the $\mathfrak{Z}_i$. Notice that $\mathfrak{Z}_0$ is of codimension one, since having a singular point for curves of class $(3,3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ induces a codimension 1 condition.

For the rest, we find the dimension of the general fiber of Φ on $\mathfrak{Y}_i$. We first treat the cases of irreducible S. Given a generic $(S, D) \in \mathfrak{Z}_2$, we obtain the Tschirnhausen embedding $D \subset \mathbb{F}_2$ by taking the minimal resolution of the A_1 singularity of S. Similarly, for a general $(S, D) \in \mathfrak{Z}_4$, we can obtain the Tschirnhausen embedding $D \subset \mathbb{F}_4$ by undoing the transformation described in § 6.2. To do so, we first take the minimal resolution of S and contract one of the two -1 curves on the resolution. Therefore, we get that $\mathfrak{Z}_2$ and $\mathfrak{Z}_4$ are of codimension 1.

We now consider the cases of reducible S. By considering the two components separately, we can reconstruct the Tschirnhausen embedding from a general (S, D) in $\mathfrak{Z}_{1/3,1/3}$ and $\mathfrak{Z}_{3,3}$, up to finitely many choices. Therefore, we get that $\mathfrak{Z}_{1/3,1/3}$ and $\mathfrak{Z}_{3,3}$ are of codimension 1.

Let us now look at the two exceptional cases. For a general $(S, D) \in \mathfrak{Z}_{1,1}$, we have $S = S_1 \cup S_2$ where both components are isomorphic to $\mathbb{P}^2$. To construct the Tschirnhausen surface $\mathbb{F}_1 \cup \mathbb{F}_1$ from S, we need to choose two points $p, q \in D := S_1 \cap S_2$ to blow up

on S_1 and S_2 respectively (the curve in $\mathbb{F}_1 \cup \mathbb{F}_1$ is simply the pre-image of D). Since p and q can be any points in $D \cong \mathbb{P}^1$, a general fiber of $\Phi \colon \mathfrak{Y}_{1,1} \to \mathfrak{Z}_{1,1}$ has dimension 2. Therefore, $\mathfrak{Z}_{1,1}$ is of codimension 3.

For the a general $(S,D) \in \mathfrak{Z}_{1,3}$, we have $S = S_1 \cup S_2$ where $S_1 \cong \mathbb{P}^2$ and S_2 is the cone over twisted cubic. To construct the Tschirnhausen surface $\mathbb{F}_1 \cup \mathbb{F}_3$, we must choose a point on the double curve of $S_1 \cup S_2$ to blow up on S_1 (see Remark 6.13). Hence, a general fiber of $\Phi \colon \mathfrak{Y}_{1,3} \to \mathfrak{Z}_{1,3}$ has dimension 1. Therefore, $\mathfrak{Z}_{1,3}$ is of codimension 2. $\square$

Although the indices $i \in I$ correspond bijectively with the divisors $\mathfrak{Y}_i$, when we pass to the $\mathfrak{Z}_i$, we have one coincidence.

Proposition 8.8. *For indices $i \neq j$ in I, we have $\mathfrak{Z}_i \neq \mathfrak{Z}_j$ Nonempty $\mathfrak{Z}_a$'s and $\mathfrak{Z}_{b,c}$'s are distinct except $\mathfrak{Z}_{3,3} = \mathfrak{Z}_{\frac{1}{3},\frac{1}{3}}$ in $\mathfrak{X}$.*

Proof. For $i \neq j$ in $I \setminus \{(3,3),(1/3,1/3)\}$, the surfaces parametrized by the general points of $\mathfrak{Z}_i$ and $\mathfrak{Z}_j$ are non-isomorphic, as they have non-isomorphic singularities. That $\mathfrak{Z}_{3,3}$ is the same as $\mathfrak{Z}_{(1/3,1/3)}$ follows from Proposition 6.8. $\square$

Proposition 8.8 shows that boundary of $\mathfrak{X}$ has 4 divisorial components, namely $\mathfrak{Z}_0$, $\mathfrak{Z}_2$, $\mathfrak{Z}_4$ and $\mathfrak{Z}_{3,3}$. The next proposition shows that they cover the entire boundary.

Proposition 8.9. *$\mathfrak{Z}_{1,1}$ is contained in $\mathfrak{Z}_2$ and $\mathfrak{Z}_{1,3}$ is contained in $\mathfrak{Z}_4$. Therefore, the boundary of $\mathfrak{X}$ is divisorial.*

Proof. Let us first consider the case of $\mathfrak{Z}_2$ and $\mathfrak{Z}_{1,1}$. Recall that a general $(S,D) \in \mathfrak{Z}_{1,1}$ arises as the stabilization of a Tschirnhausen pair $(\mathbb{F}_1 \cup \mathbb{F}_1, D)$, and a general (S,D) in $\mathfrak{Z}_2$ as the stabilization of a Tschirnhausen pair $(\mathbb{F}_2, D)$. We construct a family of Tschirnhausen pairs with generic fiber $\mathbb{F}_2$ degenerating to $\mathbb{F}_1 \cup \mathbb{F}_1$.

Take a one parameter family $\pi \colon B \to \Delta$ of smooth $\mathbb{P}^1$'s degenerating to a nodal rational curve $P_1 \cup_p P_2$ over a DVR Δ. Call $0 \in \Delta$ the special point and $\eta \in \Delta$ the generic point. Then $B_\eta \cong \mathbb{P}^1$ and $B_0 = P_1 \cup_p P_2$. Choose sections $s_i \colon \Delta \to B$ of π for $i = 1,2$ with

$s_i(0) \in P_i \setminus \{p\}$ for $i = 1, 2$. Consider the vector bundle

$$E = \mathcal{O}(s_1 + s_2) \oplus \mathcal{O}(2s_1 + 2s_2)$$

over B. Notice that the generic fiber E_η is $\mathcal{O}(2) \oplus \mathcal{O}(4)$ and the special fiber E_0 is $\mathcal{O}(1,1) \oplus \mathcal{O}(2,2)$; here $\mathcal{O}(a, b)$ denotes the line bundle on $P_1 \cup P_2$ of degree a on P_1 and degree b on P_2. Let $\mathcal{D} \subset \mathbb{P}E$ be a general divisor of class $\pi^* \mathcal{O}(3) \otimes \pi^* \det E^\vee$. We can check that

$$h^0(\pi^* \mathcal{O}(3) \otimes \pi^* \det E^\vee|_\eta) = h^0(\pi^* \mathcal{O}(3) \otimes \pi^* \det E^\vee|_0),$$

so the divisor $\mathcal{D}_0 \subset \mathbb{P}(E_0)$ is general in its linear series. The covering $\mathcal{D} \to B$ over Δ gives a map $\mu \colon \Delta \to \overline{\mathcal{H}}_4^3(1/6 + \epsilon)$. The composite $\Phi \circ \mu \colon \Delta \to \mathfrak{X}$ maps 0 to a general point of $\mathfrak{Z}_{1,1}$ and η to a point of $\mathfrak{Z}_2$. It follows that $\mathfrak{Z}_{1,1}$ is contained in $\mathfrak{Z}_2$.

The case of $\mathfrak{Z}_{1,3}$ and $\mathfrak{Z}_4$ is proved similarly by taking $E = \mathcal{O}(s_1) \oplus \mathcal{O}(2s_1 + 3s_2)$. $\qquad\square$

8.3. Comparison of $\mathfrak{X}$ with $\mathcal{M}_4$.

In this section, we describe the relationship between $\mathfrak{X}$ and the moduli stack $\mathcal{M}_4$ of smooth curves of genus 4.

Denote by $\mathfrak{M}_4$ the (non-separated) moduli stack of all curves (proper, connected, reduced schemes of dimension 1) of arithmetic genus 4. We have a forgetful map $\mathfrak{X} \to \mathfrak{M}_4$ that sends (S, D) to D. Let $\mathfrak{X}_0 \subset \mathfrak{X}$ be the open subset where D is smooth. Corollary 6.17 describes the surfaces appearing on $\mathfrak{X}_0$. The forgetful map restricts to a map

$$F \colon \mathfrak{X}_0 \to \mathcal{M}_4.$$

Proposition 8.10. *F is (1) representable, (2) proper, and (3) restricts to an isomorphism*

$$F \colon \mathfrak{X}_0 \setminus \mathfrak{Z}_4 \to \mathcal{M}_4 \setminus \mathcal{H}_4,$$

where $\mathcal{H}_4 \subset \mathcal{M}_4$ is the hyperelliptic locus.

Proof. (1) By [4, Lemma 4.4.3], it suffices to show that $F\colon \mathfrak{X}_0(\mathbb{K}) \to \mathcal{M}_4(\mathbb{K})$ is a faithful map of groupoids. In other words, given any $(S, C) \in \mathfrak{X}_0(\mathbb{K})$, we need to show that any automorphism f of (S, C) restricting to identity on C is the identity on S. We break this into two cases.

In the first case, suppose C is not hyperelliptic. Then C has a canonical embedding $C \subset \mathbb{P}^3$. The linear series $|K_S + C|$ gives an embedding of S in $\mathbb{P}^3$ as a quadric surface. So S is realized as the unique quadric surface in $\mathbb{P}^3$ containing C. Note that every automorphism of S extends uniquely to an automorphism of $\mathbb{P}^3$. That is, we have an injection

$$\mathrm{Aut}(S) \subset \mathrm{PGL}_4(\mathbb{K}).$$

Likewise, every automorphism of C extends uniquely to an automorphism of $\mathbb{P}^3$, so we also have an injection

$$\mathrm{Aut}(C) \subset \mathrm{PGL}_4(\mathbb{K}).$$

It follows that every automorphism of S that is the identity on C is the identity on S.

In the second case, suppose C is hyperelliptic. Let $\widetilde{S} \to S$ be the minimal desingularization of S. Recall that S has a $\frac{1}{9}(1, 2)$ singularity and possibly an additional A_1 singularity. The map $\widetilde{S} \to S$ resolves the $\frac{1}{9}(1, 2)$ singularity to produce a chain of rational curves of self-intersection $(-5, -2)$. We have a unique fibration $\widetilde{S} \to \mathbb{P}^1$ whose generic fiber is $\mathbb{P}^1$. The -5 curve σ obtained in the resolution is a section of this fibration. An automorphism f of S induces an automorphism $\widetilde{f}$ of $\widetilde{S}$. Note that $\widetilde{f}$ must preserve the fibration $\widetilde{S} \to \mathbb{P}^1$ and the section σ. If f also fixes C, then $\widetilde{f}$ fixes three points in a generic fiber of $\widetilde{S} \to \mathbb{P}^1$, namely the point of σ, and the two points of C. It follows that f is the identity on S.

(2) Since $\mathfrak{X}_0$ is separated and of finite type, so is F. For properness, we check the valuative criterion. Let $\pi\colon C \to \Delta$ be a smooth proper curve of genus 4. We may assume that the generic fiber C_η is non-hyperelliptic and Maroni general. Let (S_η, C_η) be an

object of $\mathfrak{X}$ over the generic point η. We must extend it to an object of $\mathfrak{X}$ over Δ that gives $C \to \Delta$ under the map F.

Since C_η is a smooth, non-hyperelliptic curve, S_η is the unique quadric surface containing C_η in its canonical embedding. Possibly after a base change on Δ, we have a line bundle L on C such that for all $t \in \Delta$, we have $\deg L_t = 3$ and $h^0(L_t) = 2$. If the central fiber C_0 is non-hyperelliptic, then L_0 is base-point free. In that case, we have a finite, flat, degree 3 map

$$f : C \to \mathbb{P}^1_\Delta = \mathbb{P}(\pi_* L).$$

If C_0 is hyperelliptic, then L_0 is given by the hyperelliptic line bundle twisted by $\mathcal{O}(p)$ for some $p \in C_0$ and has a base point at p. After finitely many blow-ups and contractions of (-2) curves centered on p, we obtain a family $\pi' : C' \to \Delta$ and a finite, flat, degree 3 map

$$f : C' \to \mathbb{P}^1_\Delta = \mathbb{P}(\pi_* L).$$

The central fiber of $C' \to \Delta$ is the nodal union of C_0 and $\mathbb{P}^1$ at p. In either case, f yields a map $\Delta \to \overline{\mathcal{H}}^3_4(1/6+\epsilon)$. Its composition with $\Phi : \overline{\mathcal{H}}^3_4(1/6+\epsilon) \to \mathfrak{X}$ gives a map $\Delta \to \mathfrak{X}$. From the description of stabilization for the central fiber of f (see § 6.1 and § 6.2), we see that Δ maps to $\mathfrak{X}_0$ and provides the necessary extension of $\eta \to \mathfrak{X}$ given by (S_η, C_η).

(3) Let $p : \operatorname{spec} \mathbb{K} \to \mathcal{M}_4 \setminus \mathcal{H}_4$ be a point given by a smooth, non-hyperelliptic curve C. The fiber of

$$(8.1) \qquad\qquad F : \mathfrak{X}_0 \setminus \mathfrak{Z}_4 \to \mathcal{M}_4 \setminus \mathcal{H}_4$$

over p is a unique point, represented by the isomorphism class of (S, C) where S is the unique quadric surface containing the canonical image of C. By Zariski's main theorem, we conclude that (8.1) is an isomorphism. $\qquad\square$

Using Proposition 8.10, we immediately deduce the following.

Theorem 8.11. *The map F induces an isomorphism of stacks*

$$\mathfrak{X}_0 \xrightarrow{\sim} \mathrm{Bl}_{\mathcal{H}_4} \mathcal{M}_4.$$

Proof. It suffices to check the statement étale locally on $\mathcal{M}_4$. So let U be a scheme and $U \to \mathcal{M}_4$ an étale map. Let $H \subset U$ be the pre-image of $\mathcal{H}_4$. Likewise, let $X \to U$ be the pullback of $\mathfrak{X}_0 \to \mathcal{M}_4$ and $Z \subset X$ the pre-image of $\mathcal{Z}_4$. Note that U and H are smooth. We may assume that they are also connected (hence irreducible).

Let p be a point of H whose image in $\mathcal{H}_4$ corresponds to the hyperelliptic curve C. By Corollary 6.17, the (set-theoretic) fiber of $Z \to H$ over p is $\mathbb{P}^1$, given by the elements of the hyperelliptic linear series on C. Since H is irreducible, and the fibers of $Z \to H$ are irreducible of the same dimension, Z is also irreducible. We also know that $X \to U$ is an isomorphism over the complement of H. Since $H \subset U$ is smooth, X is smooth, and Z is irreducible, $X \to U$ is the blow-up at H by [38, Corollary]. $\square$

Using Proposition 8.10, we also obtain the Picard group of $\mathfrak{X}$.

Proposition 8.12. *The rational Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathfrak{X})$ of $\mathfrak{X}$ is of rank 4, and is generated by the classes of the four boundary divisors.*

Proof. We have a surjective map

$$\mathrm{Pic}_{\mathbb{Q}}(\mathfrak{X}) \to \mathrm{Pic}_{\mathbb{Q}}(\mathfrak{X}_0)$$

given by pull-back, whose kernel is generated by the irreducible components of $\mathfrak{X} \setminus \mathfrak{X}_0$, namely $\mathfrak{Z}_0$ and $\mathfrak{Z}_{3,3}$. Since $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_4) = \langle \lambda \rangle$ and $\mathcal{H}_4 \subset \mathcal{M}_4$ is of codimension 2, we have

$$\mathrm{Pic}_{\mathbb{Q}}(\mathfrak{X}_0 \setminus \mathfrak{Z}_4) = \mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_4 \setminus \mathcal{H}_4) = \mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_4) = \mathbb{Q}\langle \lambda \rangle.$$

The image of $\mathfrak{Z}_2$ in $\mathcal{M}_4$ is the Maroni divisor, which is linearly equivalent to a rational multiple of λ (precisely, $\mathfrak{Z}_2 \sim 17\lambda$ by [39, Theorem IV]). Therefore, we get

$$\mathrm{Pic}_{\mathbb{Q}}\big(\mathfrak{X}_0 \setminus (\mathfrak{Z}_2 \cup \mathfrak{Z}_4)\big) = 0.$$

Hence, $\mathrm{Pic}_{\mathbb{Q}}(\mathfrak{X})$ is generated by $\mathfrak{Z}_0, \mathfrak{Z}_2, \mathfrak{Z}_4$, and $\mathfrak{Z}_{3,3}$.

We now show that the 4 boundary divisors are linearly independent by test-curve calculations. Take 3 curves C_1, C_2, C_3 in $\mathfrak{X}$ as follows:

$$C_1 := \text{a pencil of } (3,3) \text{ curves in } \mathbb{P}^1 \times \mathbb{P}^1$$

$$C_2 := \text{a curve meeting } \mathfrak{Z}_{3,3} \text{ in } \mathfrak{X}$$

$$C_3 := \text{a curve in the exceptional locus of } \mathrm{Bl}_{\mathcal{H}_4}\mathcal{M}_4 \to \mathcal{M}_4$$

The intersection matrix of C_1, C_2, C_3 and $\mathfrak{Z}_0, \mathfrak{Z}_{3,3}, \mathfrak{Z}_4$ is as follows, where $*$ denotes a non-zero number and ? an unknown number.

	$\mathfrak{Z}_0$	$\mathfrak{Z}_{3,3}$	$\mathfrak{Z}_4$
C_1	34	0	0
C_2	?	$*$	?
C_3	0	0	-1

Since this matrix is invertible, we conclude that $\mathfrak{Z}_0, \mathfrak{Z}_{3,3}, \mathfrak{Z}_4$ are linearly independent. It remains to show that $\mathfrak{Z}_2$ is linearly independent of these three. If $\mathfrak{Z}_2$ were a linear combination of $\mathfrak{Z}_0, \mathfrak{Z}_{3,3}$, and $\mathfrak{Z}_4$, then its restriction to $\mathfrak{X}_0$ would be a rational multiple of $\mathfrak{Z}_4$. But $\mathfrak{Z}_2$ and $\mathfrak{Z}_4$ are clearly linearly independent on $\mathfrak{X}_0 = \mathrm{Bl}_{\mathcal{H}_4}\mathcal{M}_4$. Indeed, $\mathfrak{Z}_4$ is the exceptional divisor of the blow up and $\mathfrak{Z}_2$ is the pullback of a non-trivial divisor on $\mathcal{M}_4$. $\qquad\square$

Theorem 8.11 implies that $\mathfrak{X}$ is a compactification of $\mathrm{Bl}_{\mathcal{H}_4}\mathcal{M}_4$. We may ask whether $\mathfrak{X}$ is the blow up of the closure of $\mathcal{H}_4$ in $\overline{\mathcal{M}}_4$. The answer is "No." In fact, we can see that F does not even extend to a morphism from $\mathfrak{X}$ to $\overline{\mathcal{M}}_4$.

To see this, observe that there is a stable log quadric $(\mathbb{P}^1 \times \mathbb{P}^1, C)$ where C is an irreducible curve with a cuspidal singularity. Let $p \in \mathfrak{X}$ be the point represented by this stable log quadric. Then the rational map $F \colon \mathfrak{X} \dashrightarrow \overline{M}_4$ is undefined at p. Let $C \to \Delta$ be a one parameter family of $(3,3)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$ with central fiber C and smooth general fiber. The stable limit of such a family in $\overline{M}_4$ is $C^\nu \cup E$, where C^ν is the normalization of C and E is an elliptic curve attached nodally to C^ν at the pre-image of the cusp. Furthermore, it is easy to see that we obtain all possible elliptic curves E by making different choices of the one parameter family Δ. Hence, it is impossible to define F at p.

The next natural question is whether there is a map from $\mathfrak{X}$ to an existing alternative compactification of $\mathcal{M}_4$? Let us consider the alternative compactifications of $\mathcal{M}_4$ constructed in the Hassett–Keel program [15], which we now recall. Let $\alpha \in [0,1]$ be such that $K_{\overline{\mathcal{M}}_4} + \alpha\delta$ is effective (here δ is the class of the boundary divisor of $\overline{\mathcal{M}}_4$), we have the space

$$\overline{M}_4(\alpha) = \mathrm{Proj} \bigoplus_{m \geq 0} H^0\left(\overline{\mathcal{M}}_4, m(K_{\overline{\mathcal{M}}_4} + \alpha\delta)\right).$$

We restrict ourselves to $\alpha > 2/3 - \epsilon$ for a small enough ϵ. For such α, the spaces $\overline{M}_4(\alpha)$ can be described as the good moduli spaces of various open substacks of the stack of all curves $\mathfrak{M}_4$ [7]. The answer, however, still turns out to be negative.

Proposition 8.13. *For any value of $\alpha \in (2/3 - \epsilon, 1] \cap \mathbb{Q}$, the map F does not extend to a morphism from $\mathfrak{X}$ to $\overline{M}_4(\alpha)$.*

Proof. There is a stable log quadric $(\mathbb{P}^1 \times \mathbb{P}^1, C)$ where C is irreducible with an A_4 (rhamphoid cusp) singularity. Let p be the point of $\mathfrak{X}$ corresponding to $(\mathbb{P}^1 \times \mathbb{P}^1, C)$. But $\overline{M}_4(\alpha)$ contains a point representing a curve with a rhamphoid cusp only if $\alpha \leq 2/3$.

We conclude that for $\alpha > 2/3$, the rational map $F \colon \mathfrak{X} \dashrightarrow \overline{M}_4(\alpha)$ must be undefined at p. Indeed, for $\alpha > 2/3$, the limit in $\overline{M}_4(\alpha)$ of a one parameter family of generically smooth $(3,3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ curves limiting to C is $C^\nu \cup T$, where C^ν is the normalization of C and T is a genus 2 curve attached to C^ν at the pre-image of the rhamphoid cusp on C and at a Weierstrass point of T [23, 6.2.2]. Furthermore, we can see that multiple Weierstrass genus 2 tails T arise (in fact, all of them do) by different choices of the family. So F cannot be defined at p.

It remains to show that F does not extend to a map to $\overline{M}_4(\alpha)$ for $2/3 - \epsilon < \alpha \leq 2/3$. The culprit here is the locus $\mathfrak{Z}_{1,3}$. Let $p \in \mathfrak{X}$ be a generic point of $\mathfrak{Z}_{1,3}$. Recall that the curve in the pair corresponding to p is a genus 2 curve with an elliptic bridge. We will show that the elliptic bridge causes $F \colon \mathfrak{X} \dashrightarrow \overline{M}_4(\alpha)$ to be undefined at p. On one hand, p lies in the closure of the hyperelliptic locus $\mathfrak{Z}_4$ by Proposition 8.9. Therefore, if F is defined at p, then $F(p)$ must lie in the closure of the hyperelliptic locus in $\overline{M}_4(\alpha)$. On the other hand, we construct a one parameter family $\Delta \to \mathfrak{X}$ with central fiber p whose stable limit in $\overline{M}_4(\alpha)$ does not lie in the closure of the hyperelliptic locus. This will show that F cannot be defined at p.

To construct Δ, start with a family $\mathcal{P} \to \Delta$ whose generic fiber $\mathcal{P}_\eta$ is $\mathbb{P}^1$, whose special fiber $\mathcal{P}_0$ is a nodal rational chain of length 2, and whose total space $\mathcal{P}$ is non-singular. Take a vector bundle $\mathcal{E}$ on $\mathcal{P}$ such that $\mathcal{E}_\eta \cong \mathcal{O}(3) \oplus \mathcal{O}(3)$ and $\mathcal{E}_0 \cong \mathcal{O}(1,0) \oplus \mathcal{O}(2,3)$. Let $\mathcal{C} \subset \mathbb{P}\mathcal{E}$ be a general divisor in the linear series $\mathcal{O}_{\mathbb{P}E}(3) \otimes \det \mathcal{E}^\vee$. Observe that the central fiber $\mathbb{P}\mathcal{E}_0$ is $\mathbb{F}_1 \cup \mathbb{F}_3$. The divisor $\mathcal{C}_0 \cap \mathbb{F}_1$ is the pre-image of a general plane cubic and is disjoint from the directrix. The divisor $\mathcal{C}_0 \cap \mathbb{F}_3$ is the disjoint union of the directrix and a hyperelliptic curve H of genus 2. The curve H meets the elliptic curve nodally at two points, say q and r, which are hyperelliptic conjugate. We have seen that the stabilization of the central fiber $(\mathbb{P}\mathcal{E}_0, \mathcal{C}_0)$ is a point of $\mathfrak{Z}_{1,3}$ (§ 6.5).

We now find the stable limit of the family $\mathcal{C} \to \Delta$ in $\overline{M}_4(\alpha)$. To do so, we must contract the rational tail and the elliptic bridge of $\mathcal{C}_0$. It will be useful to achieve this

contraction in the family of surfaces $\mathbb{P}\mathcal{E} \to \Delta$. Let $\mathcal{X}_1 \to \mathbb{P}\mathcal{E}$ be the blow up of the directrix $\sigma \subset \mathbb{F}_1 \subset \mathbb{P}\mathcal{E}_0$. From the sequence

$$0 \to \mathcal{O}(-1) = N_{\sigma/\mathbb{F}_1} \to N_{\sigma/\mathcal{X}} \to N_{\mathbb{F}_1/\mathcal{X}}\big|_\sigma = \mathcal{O}(-1) \to 0$$

we see that the normal bundle of σ in $\mathbb{P}\mathcal{E}$ is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Hence the exceptional divisor of the blow up is $\mathbb{P}^1 \times \mathbb{P}^1$ and it is disjoint from the proper transform of $\mathcal{C}$. The proper transform of $\mathbb{F}_1 \subset \mathbb{P}\mathcal{E}_0$ is a copy of $\mathbb{F}_1$. The proper transform of $\mathbb{F}_3 \subset \mathbb{P}\mathcal{E}_0$ is $\mathrm{Bl}_p \mathbb{F}_3$ where $p = \sigma \cap \mathbb{F}_3$. We contract the exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathcal{X}_1$ in the other direction, namely along the fibers opposite to the fibers of the projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \sigma$, obtaining a threefold $\mathcal{X}_2$ (this is a contraction of type Theorem 2.13(1)). The central fiber of $\mathcal{X}_2 \to \Delta$ is $\mathbb{P}^2 \cap \mathrm{Bl}_p \mathbb{F}_3$. We next contract the $\mathbb{P}^2$ in the central fiber to obtain $\mathcal{X}_3$ (this is a contraction of type Theorem 2.13(2)). The central fiber of $\mathcal{X}_3 \to \Delta$ is $\mathbb{F}_2$. On this $\mathbb{F}_2$, the central fiber of the proper transform of $\mathcal{C}$ is a divisor of class $-3/2K$. More precisely, it is the union of the directrix s and a curve of class $2s + 6f$ with a node on the directrix. Finally, let $\mathcal{X}_3 \to \mathcal{X}_4$ be the small contraction of s (obtained by taking the anti-canonical model). The central fiber of $\mathcal{X}_4 \to \Delta$ is the cone over a plane conic, namely a singular quadric surface in $\mathbb{P}^3$. Let $\mathcal{C}_4 \subset \mathcal{X}_4$ be the proper transform of $\mathcal{C}$. The central fiber C of $\mathcal{C}_4 \to \Delta$ is a tacnodal curve whose normalization is H; the pre-image of the tacnode is the hyperelliptic conjugate pair $\{q, r\}$. Most importantly, however, we have $C \subset Q$ where $Q \subset \mathbb{P}^3$ is a quadric surface. As a result, we see that C has a canonical embedding in $\mathbb{P}^3$. Therefore, it is not in the closure of the hyperelliptic locus. This observation completes the proof of the assertion that F cannot be defined at a generic point of $\mathfrak{Z}_{1,3}$. $\qquad\square$

Denote by X the coarse space of $\mathcal{X}$. Proposition 8.13 says that the relationship between X and the known modular compactifications of M_4 is complicated.

We close with some questions.

Question 8.14. *How does the birational map $X \dashrightarrow \overline{M}_4(\alpha)$ decompose into more elementary birational transformations (divisorial contractions and flips)? Is X a log canonical model of $\mathrm{Bl}_{\mathcal{H}_4} \overline{\mathcal{M}}_4$?*

Recall that X can be interpreted as the KSBA compactification of weighted pairs (S, wC) with weight $w = 2/3 + \epsilon$ for sufficiently small $\epsilon < \frac{1}{30}$. An answer to the following question will be interesting in itself, and also potentially useful for Question 8.14.

Question 8.15. *How does the KSBA compactification change as the weight w varies in $(2/3, 1]$?*

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